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INSTITUT DES HAUTES ETUDES SCIENTIFIQUES

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A FAMILY OF SIMPLE GROUPS ASSOCIATED WITH THE SIMPLE LIE ALGEBRA OF TYPE (F_4).*

By RIMHAK REE.[1]

Introduction. In [4] the author constructed a family of new simple groups by applying to the Chevalley groups of type (G_2) a method which was obtained by analyzing Suzuki groups [5]. In this paper, we obtain another family of simple groups, which also seem to be new,[2] by applying the same method to the Chevalley groups (F_4). The orders of the finite groups in the family are given by

$$q^{12}(q-1)(q^3+1)(q^4-1)(q^6+1),$$

where $q=2^{2n+1}$, $n=1,2,3,\cdots$.

The construction is based on a certain automorphism[3] of the Chevalley group of type (F_4) over a field of characteristic 2. The existence of such an automorphism is already suggested in [3, Exposé 24, pp. 4-5]. Our group turns out to be the group of invariant elements under the automorphism.

Throughout this paper, we shall use the notation of [4, Section 1].

1. The root system of type (F_4). The root system Σ of type (F_4) consists of the 48 roots:

$$\xi_i, \quad \xi_i+\xi_j, \quad \tfrac{1}{2}(\xi_i+\xi_j+\xi_k+\xi_l), \tag{1.1}$$

where: $i,j,k,l=\pm 1,\pm 2,\pm 3,\pm 4$; $|i|$, $|j|$, $|k|$, $|l|$ are distinct; $\xi_{-i}=-\xi_i$ for all i.

Let P be the additive group generated by Σ, and $V=\boldsymbol{R}\otimes P$ the vector

* Received September 13, 1960; revised February 21, 1961.

[1] This work was done while the author held a Research Associateship, 1959-1960, of the Office of the Naval Research, U. S. Navy. The content of this paper, together with that of [4], was announced in a lecture given at the American Mathematical Society Summer Institute on Finite Groups, Pasadena, California, on August 22, 1960.

The author would like to express gratitude to J. Tits for his perusal of the manuscript and many valuable suggestions.

[2] J. Tits informs the author that he has proved, by treating the orders of the finite groups by Artin's method [6], that the finite groups in our family are actually new.

[3] The author learned at Pasadena that the idea of obtaining simple groups by using this automorphism was also conceived, independently, by Robert Steinberg.

space over the real number field $\boldsymbol{R}$. Define the positive definite metric $(\xi \mid \eta)$ of V by $(\xi_i \mid \xi_i) = 1$, $(\xi_i \mid \xi_j) = 0$ for $i, j = 1, 2, 3, 4$; $i \neq j$. Then one can verify that the number $s(r)$ defined in [4, Section 1] can be written $s(r) = 2(s \mid r)/(r \mid r)$. Hence for the element w_r in the Weyl group W of Σ, we have $w_r(s) = s - (2(s \mid r)/(r \mid r))r$ for all $s \in \Sigma$. From this it follows that W leaves the metric $(\xi \mid \eta)$ invariant. By (1.1) one can see easily that $(r \mid r) = 1$ or 2, and that $(r \mid s) = 0$ or ± 1 for any $r, s \in \Sigma$. We shall write $\lambda(r) = (r \mid r)$.

We shall prove a few lemmas which will be used later.

(1.2) If $r, s, r+s \in \Sigma$, $\lambda(r) = \lambda(s) = 2$, then $\lambda(r+s) = 2$.

Proof.

$$\lambda(r+s) = (r+s \mid r+s) = (r \mid r) + 2(r \mid s) + (s \mid s) = 4 + 2(r \mid s) = 1 \text{ or } 2.$$

Then, since $(r \mid s) = 0$ or ± 1, we must have $(r \mid s) = -1$, $\lambda(r+s) = 2$.

(1.3) If $r, s, r+s \in \Sigma$, $\lambda(r) = 1$, $\lambda(s) = 2$, then $2r+s \in \Sigma$, and $\lambda(r+s) = 1$, $\lambda(2r+s) = 2$.

Proof. $\lambda(r+s) = (r+s \mid r+s) = 3 + 2(r \mid s) = 1$ or 2. Since $(r \mid s) = 0$ or ± 1, we must have $(r \mid s) = -1$, $\lambda(r+s) = 1$. Then $\lambda(r-s) = 3 - 2(r \mid s) = 5$. Hence $r - s \notin \Sigma$. Then $s(r) = 2(s \mid r)/(r \mid r) = -2$ implies that $2r + s \in \Sigma$. We have $\lambda(2r+s) = 4 + 2 + 2(r \mid s) = 2$.

Now, define the linear map $\psi: V \to V$ by

$$\psi(\xi_1) = \xi_1 + \xi_2, \quad \psi(\xi_2) = \xi_1 - \xi_2, \quad \psi(\xi_3) = \xi_3 + \xi_4, \quad \psi(\xi_4) = \xi_3 - \xi_4.$$

Then we have (cf. [3], Exposé 24, pp. 4-5)

(1.4) There exists a permutation $r \to \bar{r}$ of order 2 of Σ such that $\psi(r) = \lambda(r)\bar{r}$, $\overline{-r} = -\bar{r}$, $\lambda(r)\lambda(\bar{r}) = 2$ for all $r \in \Sigma$.

By the definition of ψ, one can see easily that $\psi = \sqrt{2}\psi_\delta$, where ψ_δ is an orthogonal transformation (in fact, a symmetry) of the vector space V. From this and (1.4) we have

(1.5) $\lambda(s)(\bar{r} \mid \bar{s}) = \lambda(\bar{r})(r \mid s)$ for all $r, s \in \Sigma$.

(1.6) $\overline{w_r(s)} = w_{\bar{r}}(\bar{s})$ for all $r, s \in \Sigma$.

Proof. This follows easily from (1.5).

(1.7) The map $w \to \bar{w} = \psi \circ w \circ \psi^{-1}$ is an automorphism of the Weyl group W such that $\overline{w_r} = w$ for all $r \in \Sigma$.

Proof. This is immediate from (1.6).

(1.8) If $r, s, r+s \in \Sigma$, and $\bar{r}+\bar{s} \notin \Sigma$, then $\lambda(r)=\lambda(s)=1, \lambda(r+s)=2$.

Proof. Since $\bar{r}+\bar{s} \notin \Sigma$, we have $\bar{s}(\bar{r}) \geqq 0$, $(\bar{r} \mid \bar{s}) \geqq 0$. Then by (1.5), $(r \mid s) \geqq 0$. Now, $(r+s \mid r+s)=(r \mid r)+2(r \mid s)+(s \mid s)=1$ or 2. Hence we must have $\lambda(r)=\lambda(s)=1$, $(r \mid s)=0$, $\lambda(r+s)=2$.

(1.9) If $r \in \Sigma, \lambda(r)=1, r+\bar{r} \in \Sigma$, then $\lambda(r+\bar{r})=1$ and $\overline{r+\bar{r}}=2r+\bar{r}$.

Proof. Since $\lambda(\bar{r})=2$ by (1.4), $\lambda(r+\bar{r})=1$ follows from (1.3). Then $r+\bar{r}=\psi(r+\bar{r})=\bar{r}+2r$.

(1.10)[4] The group W^1 of all $w \in W$ such that $\bar{w}=w$ is of order 16, and consists of $w(\epsilon, \delta, \infty)$, $w(\infty, \epsilon, \delta)$, and $w(\epsilon, \delta, \eta)$, $\epsilon, \delta, \eta = \pm 1$, which are defined by:

$w(\epsilon, \delta, \infty)$:

$$\xi_1 \to \epsilon\xi_1,\ \xi_2 \to \epsilon\xi_2,\ \xi_3 \to \delta\xi_3,\ \xi_4 \to \delta\xi_4;$$

$w(\infty, \epsilon, \delta)$:

$$\xi_1 \to \epsilon\xi_3,\ \xi_2 \to \epsilon\xi_4,\ \xi_3 \to \delta\xi_1,\ \xi_4 \to \delta\xi_2;$$

$w(\epsilon, \delta, \eta)$:

$$\xi_1 \to \tfrac{1}{2}(\epsilon(\xi_1+\xi_2)+\delta(\xi_3+\xi_4)),$$
$$\xi_2 \to \tfrac{1}{2}(\epsilon(\xi_1-\xi_2)+\delta(\xi_3-\xi_4)),$$
$$\xi_3 \to \tfrac{1}{2}\eta(\epsilon(\xi_1+\xi_2)-\delta(\xi_3+\xi_4)),$$
$$\xi_4 \to \tfrac{1}{2}\eta(\epsilon(\xi_1-\xi_2)-\delta(\xi_3-\xi_4)).$$

Proof. Let $\bar{w}=w$. Since $\lambda(w\xi_i)=1$ for all i, we have $w\xi_1=\pm\xi_i$ or $\frac{1}{2}\Sigma\epsilon_i\xi_i$, where $\epsilon_i=\pm 1$. If $w\xi_1=\pm\xi_i$, then the property $(w\xi_1 \mid w\xi_i)=0$ for $i \neq 1$ shows that all $w\xi_i$ are of the form $\pm\xi_i$. Then by using the condition $\psi \circ w = w \circ \psi$ one deduces easily that $w=w(\epsilon, \delta, \infty)$ or $w(\infty, \epsilon, \delta)$ given above. If $w\xi_1$ is of the form $\frac{1}{2}\Sigma\epsilon_i\xi_i$, the same argument as above shows that all $w\xi_i$ have the same form, and from this one can deduce $w=w(\epsilon, \delta, \eta)$.

2. An automorphism of the Chevalley group. The basis elements of the simple Lie algebra of type (F_4) given by Cartan [1, p. 224] can be modified, by multiplying the coefficients ± 1 suitably, so that they satisfy the conditions in Theorem 1 of [2, p. 24]. If, however, the ground field K is of characteristic 2, then Cartan's basis elements can be used directly to define the Chevalley group G of type (F_4) over K (see [4, Section 1] for a definition).

We shall write $x_i(t)$, $x_{ij}(t)$, $x_{ijkl}(t)$ to denote the elements $x_r(t)$, where

[4] The group W^1 is actually dihedral.

r are, respectively, the roots given in (1.1). In this notation, we shall write i' instead of $-i$. Thus, for example, $x_{ijk'l'}(t)$ denotes $x_r(t)$, where $r=\frac{1}{2}(\xi_i+\xi_j-\xi_k-\xi_l)$. This way of indicating roots which appear as subscripts will be applied to other symbols also.

By the multiplication table given by Cartan [1, p. 224] and the formulas given by Chevalley [2, p. 26] one obtains the following expressions for the commutators (assuming that K is of characteristic 2):

$$
\begin{aligned}
&(x_i(t), x_j(u)) = 1,\\
&(x_i(t), x_{i'j}(u)) = x_j(tu)x_{ij}(t^2u),\\
&(x_{ij}(t), x_{j'k}(u)) = x_{ik}(tu),\\
&(x_i(t), x_{i'jkl}(u)) = x_{ijkl}(tu),\\
&(x_{ij}(t), x_{i'j'kl}(u)) = x_{ijkl}(tu)x_{kl}(tu^2),\\
&(x_{ijkl}(t), x_{ij'k'l'}(u)) = x_i(tu),\\
&(x_{ijkl}(t), x_{ijk'l'}(u)) = 1,
\end{aligned}
\tag{2.1}
$$

where $(x,y)=x^{-1}y^{-1}xy$, and where all the other commutators of the form $(x_r(t), x_s(u))$ are 1.

It is easy to see that (2.1), above, is equivalent to (2.2)-(2.5), below:

(2.2) If $r, s \in \Sigma$, $r+s \notin \Sigma$, then $(x_r(t), x_s(u)) = 1$.

(2.3) If $r, s, r+s \in \Sigma$, $\lambda(r)=\lambda(s)=1$, then $(x_r(t), x_s(u)) = x_{r+s}(tu)$ or 1, according as $\lambda(r+s)=1$ or 2.

(2.4) If $r, s, r+s \in \Sigma$, $\lambda(r)=\lambda(s)=2$, then $(x_r(t), x_s(u)) = x_{r+s}(tu)$.

(2.5) If $r, s, r+s \in \Sigma$, $\lambda(r)=1$, $\lambda(s)=2$, then
$$(x_r(t), x_s(u)) = x_{r+s}(tu)x_{2r+s}(t^2u).$$

Now we shall prove

(2.6) Let $t \to t^\theta$ be an automorphism of the field K.

Then, in (2.1), we can replace $x_r(t)$ by

$$\bar{x}_r(t) = x_{\bar{r}}(t^{\lambda(\bar{r})\theta})$$

without losing the validity of the statements.

Proof. Denote by $(2.n')$, $2 \leqq n \leqq 5$, the statement obtained from $(2.n)$ by the replacement $x_r(t) \to \bar{x}_r(t)$. It suffices to prove that (2.2')-(2.5') are valid.

We shall prove (2.2'). If $\bar{r}+\bar{s} \notin \Sigma$, then (2.2') follows immediately from (2.2). If $\bar{r}+\bar{s} \in \Sigma$, then by (1.8) (using $\bar{r}$, $\bar{s}$ instead of r, s) we have

$\lambda(\bar{r}) = \lambda(\bar{s}) = 1$, $\lambda(\bar{r} + \bar{s}) = 2$. Hence by (2.3) we have $(x_{\bar{r}}(t), x_{\bar{s}}(u)) = 1$ from which (2.2′) follows at once.

We shall prove (2.3′). If $\lambda(r+s) = 1$, then $\bar{r} + \bar{s} = \psi(r+s) = \overline{r+s} \in \Sigma$. We have $\lambda(\bar{r}) = \lambda(\bar{s}) = 2$. Hence by (2.4),

$$\begin{aligned}(\bar{x}_r(t), \bar{x}_s(u)) &= (x_{\bar{r}}(t^{2\theta}), x_s(u^{2\theta})) \\ &= x_{\overline{r+s}}((tu)^{2\theta}) = \bar{x}_{r+s}(tu).\end{aligned}$$

If $\lambda(r+s) = 2$, then $\bar{r} + \bar{s} = \psi(r+s) = 2(\overline{r+s}) \notin \Sigma$. Hence we obtain $(\bar{x}_r(t), \bar{x}_s(u)) = 1$ by (2.2).

We shall prove (2.4′). We have $\lambda(r+s) = 2$ by (1.2). Hence $2\bar{r} + 2\bar{s} = \psi(r+s) = 2(\overline{r+s})$, $\bar{r} + \bar{s} = \overline{r+s} \in \Sigma$, $\lambda(\bar{r} + \bar{s}) = 1$. Then by (2.3) we have

$$(\bar{x}_r(t), \bar{x}_s(u)) = (x_{\bar{r}}(t^{\theta}), x_s(u^{\theta})) = x_{\overline{r+s}}((tu)^{\theta}) = \bar{x}_{r+s}(tu).$$

We shall prove (2.5′). By (1.3) we have $\lambda(r+s) = 1$, $\lambda(2r+s) = 2$. Hence $2\bar{r} + 2\bar{s} = \psi(2r+s) = 2(\overline{2r+s})$. Thus $\bar{r} + \bar{s} = \overline{2r+s} \in \Sigma$. Also, $\bar{r} + 2\bar{s} = \overline{r+s}$. Hence

$$\begin{aligned}(\bar{x}_r(t), \bar{x}_s(u)) &= (x_{\bar{r}}(t^{2\theta}), x_s(u^{\theta})) = (x_s(u^{\theta}), x_{\bar{r}}(t^{2\theta}))^{-1} \\ &= (x_{s+\bar{r}}(u^{\theta}t^{2\theta})\, x_{2s+\bar{r}}(u^{2\theta}t^{2\theta}))^{-1} \\ &= (\bar{x}_{2r+s}(ut^2)\, \bar{x}_{r+s}(ut))^{-1} \\ &= \bar{x}_{r+s}(tu)\, \bar{x}_{2r+s}(t^2u).\end{aligned}$$

Thus the proof of (2.6) is complete.

Using (2.6), above, and proceeding as in [4], one can prove

THEOREM 2.7.[5] *If K is a perfect field of characteristic 2, and if $t \to t^{\theta}$ is an automorphism of K, then the Chevalley group G of type (F_4) over K admits an automorphism $x \to x^{\sigma}$ such that*

$$x_r(t)^{\sigma} = x_{\bar{r}}(t^{\lambda(\bar{r})\theta})$$

for all $r \in \Sigma$ and $t \in K$. We have

$$h(\chi)^{\sigma} = h(\chi), \qquad \omega_r{}^{\sigma} = \omega_{\bar{r}}$$

for all $\chi \in X$ and $r \in \Sigma$, where $\tilde{\chi}$ is defined by

$$\tilde{\chi}(r) = \chi(\psi(r))^{\theta} = \chi(\bar{r})^{\lambda(r)\theta}.$$

---

[5] After the completion of earlier versions of this paper and [4], the author learned from Tits that the existence of the automorphism σ was already mentioned in [9], p. 282. See also R. Steinberg [8].

3. Invariant elements. From now on we assume that K is a field of characteristic 2 and that K admits an automorphism $t \to t^\theta$ such that

$$2\theta^2 = 1, \tag{3.1}$$

the right hand side being the identity automorphism.[6]

Let G be the Chevalley group of type (F_4) over K and $x \to x^\sigma$ the automorphism of G given in Theorem 2.7. An element $x \in G$ will be said to be *σ-invariant* if $x^\sigma = x$.

We shall order the vector space V (and hence Σ) lexicographically with respect to the coefficients of $\xi_1, \xi_2, \xi_3, \xi_4$. Thus $\xi = \sum c_i \xi_i > 0$ if $\xi \neq 0$ and if the first nonzero number of c_1, c_2, c_3, c_4 is positive.

Let $\mathfrak{U}$ (resp. $\mathfrak{V}$) be the subgroup of G generated by $x_r(t)$ with $r > 0$ (resp. with $r < 0$). Denote by $\mathfrak{U}^1$ (resp. $\mathfrak{V}^1$) the group of σ-invariant elements in $\mathfrak{U}$ (resp. in $\mathfrak{V}$), and by G^1 the group generated by $\mathfrak{U}^1$ and $\mathfrak{V}^1$. Denote by $\mathfrak{H}^1$ the group of all σ-invariant elements in $\mathfrak{H}$, the group of all $h(\chi)$, $\chi \in X$.

First we shall consider $\mathfrak{U}^1$. We shall denote the roots in (1.1) by i, ij, $ijkl$, respectively, writing i' for $-i$. The set of all positive roots in Σ is decomposed into the following 8 disjoint subsets:

$$\begin{aligned} E_1 &= \{2, 12', 1, 12\}, & E_5 &= \{14, 1234'\}, \\ E_2 &= \{4, 34', 3, 34\}, & E_6 &= \{14', 123'4\}, \\ E_3 &= \{12'34', 24, 1234, 13\}, & E_7 &= \{23, 12'34\}, \\ E_4 &= \{12'3'4, 24', 123'4', 13'\}, & E_8 &= \{23', 12'3'4'\}. \end{aligned} \tag{3.2}$$

The sets E_1, E_2, E_3, and E_4 are of the form $\{r, \bar{r}, r+\bar{r}, 2r+\bar{r}\}$, while E_5, E_6, E_7, and E_8 are of the form $\{r, \bar{r}\}$, where $r + \bar{r} \notin \Sigma$. Since $\overline{r+\bar{r}} = 2r + \bar{r}$, for $E_1, \cdots, E_4$, one can see easily that, for each E_i, the group generated by $x_s(t)$, $s \in E_i$, $t \in K$ is mapped onto itself by the automorphism σ of G given in (2.7). For $E_i = \{r, \bar{r}\}$, $\lambda(r) = 1$, an element $x_r(t)x_{\bar{r}}(u)$ is σ-invariant if and only if $t = u^\theta$. For $E_i = \{r, \bar{r}, r+\bar{r}, 2r+\bar{r}\}$, let x be an element in the group generated by all $x_s(t)$, $s \in E_i$, $t \in K$. Then

$$\begin{aligned} x^\sigma &= (x_r(t_1)x_{\bar{r}}(t_2)x_{r+\bar{r}}(t_3)x_{2r+\bar{r}}(t_4))^\sigma \\ &= x_{\bar{r}}({t_1}^{2\theta})x_r({t_2}^\theta)x_{2r+\bar{r}}({t_3}^{2\theta})x_{r+\bar{r}}({t_4}^\theta) \\ &= x_r({t_2}^\theta)x_{\bar{r}}({t_1}^{2\theta})x_{r+\bar{r}}({t_1}^{2\theta}{t_2}^{2\theta} + {t_4}^\theta)x_{2r+\bar{r}}({t_1}^{2\theta}{t_2}^{2\theta} + {t_3}^{2\theta}) \end{aligned}$$

by (2.5). Hence, by (3.1), $x^\sigma = x$ if and only if $t_1 = {t_2}^\theta$, $t_3 = {t_2}^{\theta+1} + {t_4}^\theta$. Thus, if x is σ-invariant, then x is written uniquely in the form

$$x_r(t^\theta)x_{\bar{r}}(t)x_{r+\bar{r}}(t^{\theta+1})x_s(u^\theta)x_s(u),$$

[6] It was discovered by Tits that K need not be assumed to be perfect if one considers an endomorphism η, instead of θ, of K such that $\eta^2 = 2$. For the details, see [10].

where $s = r + \bar{r}$, and where $t, u \in K$. The σ-invariant elements obtained from each E_i in the above manner will be called the σ-invariant elements associated with E_i.

Now we shall define the following σ-invariant elements:[7]

$$\begin{aligned}
\alpha_1(t) &= x_4(t^\theta)x_{34'}(t)x_3(t^{\theta+1}),\\
\alpha_2(t) &= x_3(t^\theta)x_{34}(t),\\
\alpha_3(t) &= x_{12'3'4'}(t^\theta)x_{23'}(t),\\
\alpha_4(t) &= x_{12'3'4}(t^\theta)x_{24'}(t)x_{123'4'}(t^{\theta+1}),\\
\alpha_5(t) &= x_2(t^\theta)x_{12'}(t)x_1(t^{\theta+1}),\\
\alpha_6(t) &= x_{12'34'}(t^\theta)x_{24}(t)_{1234}(t^{\theta+1}),\\
\alpha_7(t) &= x_{12'34}(t^\theta)x_{23}(t),\\
\alpha_8(t) &= x_{123'4'}(t^\theta)x_{13'}(t),\\
\alpha_9(t) &= x_{123'4}(t^\theta)x_{14'}(t),\\
\alpha_{10}(t) &= x_{1234'}(t^\theta)x_{14}(t),\\
\alpha_{11}(t) &= x_{1234}(t^\theta)x_{13}(t),\\
\alpha_{12}(t) &= x_1(t^\theta)x_{12}(t).
\end{aligned}$$

We shall denote by R_i the set of roots appearing in the above expression of $\alpha_i(t)$, and by $\min(R_i)$ the minimum root in R_i. Thus, for instance, $R_1 = \{\xi_4, \xi_3 - \xi_4, \xi_3\}$, and $\min(R_1) = \xi_4$. We can verify from the above that

$$\min(R_1) < \min(R_2) < \cdots < \min(R_{12}). \tag{3.3}$$

We shall prove

(3.4) If an element

$$x = x_{r_1}(t_1)x_{r_2}(t_2)\cdots x_{r_m}(t_m)$$

where all $r_i > 0$, is σ-invariant, then x can be written uniquely as

$$x = \alpha_1(u_1)\alpha_2(u_2)\cdots\alpha_{12}(u_{12})$$

with $u_i \in K$. For any i such that $u_i \neq 0$, any root in R_i is a linear combination, with integral coefficients $\geqq 0$, of the roots $r_1, r_2, \cdots, r_m$.

Proof. By (1.2) of [4], we can assume without loss of generality that $r_1 < r_2 < \cdots < r_m$, and that all $t_i \neq 0$. Since x is σ-invariant,

$$x = x_{\bar{r}_1}(\bar{t}_1)x_{\bar{r}_2}(\bar{t}_2)\cdots x_{\bar{r}_m}(\bar{t}_m),$$

where, for all i,

$$\bar{t}_i = t_i^{\lambda(\bar{r}_i)\theta}. \tag{3.5}$$

[7] For an idea of avoiding the use of the elements $\alpha_1(t), \cdots, \alpha_{12}(t)$, see [10, section 2]. The conjecture stated there was proved by Tits himself.

Let $\bar{r}_i$ be the minimum root in the set $\{\bar{r}_1, \bar{r}_2, \cdots, \bar{r}_m\}$. Then clearly x can be written in the form

$$x = x_{\bar{r}_i}(\bar{t}_i)x',$$

where x' is a product of elements $x_r(t)$ with $r > \bar{r}_i$. Then by Lemma 6 of [2, p. 39] we have

(3.6) $$r_1 = \bar{r}_i, \qquad t_1 = \bar{t}_i.$$

We can clearly write

$$x = x_{r_1}(t_1)x_{r_i}(t_i)x'',$$

where x'' is a product of the elements $x_r(t)$ such that r is a linear combination, with integral coefficients $\geqq 0$, of the roots $r_1, r_2, \cdots, r_m$, and such that $r > r_1$. Now (3.5) and (3.6) show that there exists an $\alpha_k(v)$ such that $x = \alpha_k(v)y$, where y is a product of the elements $x_r(t)$ such that r is a linear combination, with integral coefficients $\geqq 0$, of the roots $r_1, \cdots, r_m$ and such that $r > r_1$. Now we can proceed by induction on r_1, which is clearly uniquely determined by x. If r_1 is sufficiently large, then (3.4) is clear. Hence by applying the induction assumption to y, we have

$$y = \alpha_{k_1}(u_1) \cdots \alpha_{k_n}(u_n),$$

where $k_1 < \cdots < k_n$. Clearly $k < k_1$ (if $u_1 \neq 0$). Hence we conclude that x can be written in the desired form.

The uniqueness part follows easily from Lemma 6 of [2, p. 39] and (3.3), above. The rest of (3.4) is also clear.

(3.7) For any i, j, the commutator $(\alpha_i(t), \alpha_j(u))$ can be written as

$$(\alpha_i(t), \alpha_j(u)) = \alpha_k(v_k)\alpha_{k+1}(v_{k+1}) \cdots \alpha_{12}(v_{12}),$$

where $\min(R_k) \geqq \min(R_i) + \min(R_j)$.

Proof. By (1.2) of [4] one can see easily that the commutator $(\alpha_i(t), \alpha_j(u))$ can be written as $\prod x_r(t_r)$, where $r \geqq \min(R_i) + \min(R_j)$. Hence (3.7) follows immediately from (3.4).

Denote by $\mathfrak{U}^i$ the group generated by $\alpha_i(t), \alpha_{i+1}(t), \cdots, \alpha_{12}(t)$, $t \in K$. Note that, by (3.4), the new definition for $\mathfrak{U}^1$ is equivalent to the old one given at the beginning of this section. Also, set $\mathfrak{U}^{13} = \{1\}$. Then from (3.7) one can derive

(3.8) Let $1 \leqq i \leqq 12$. If $x \in \mathfrak{U}^1$, $y \in \mathfrak{U}^i$, then $(x, y) \in \mathfrak{U}^{i+1}$.

For any element $w \in W$, define $\mathfrak{U}_w$ to be the subgroup of $\mathfrak{U}$ generated by all $x_r(t)$ such that $w(r) < 0$. If $w \in W^1$ (see (1.10)), and if $w(r) < 0$

for some $r \in R_i$, then by (1.7) we have $w(r) < 0$ for all $r \in R_i$. In this case, we shall write $w(R_i) < 0$. We have

(3.9) If $w \in W^1$, then every element in $\mathfrak{U}_w \cap \mathfrak{U}^1$ is written uniquely as $x = \alpha_1(t_1)\alpha_2(t_2)\cdots\alpha_{12}(t_{12})$, where $t_i = 0$ unless $w(R_i) < 0$.

Proof. This also follows immediately from (3.4).

Now we shall consider the group $\mathfrak{B}^1$. By considering negative roots instead of positive roots, one obtains, as generators of $\mathfrak{B}^1$, the elements $\alpha'_1(t), \alpha'_2(t), \cdots, \alpha'_{12}(t)$, where $\alpha'_i(t)$ is the element obtained by replacing every x_r by x_{-r} in the expression of $\alpha_i(t)$. Thus, for instance,

$$\alpha'_1(t) = x_{4'}(t^\theta)x_{3'4}(t)x_{3'}(t^{\theta+1}).$$

We shall prove

(3.10) Let $E_i = \{r, \bar{r}, r+r, 2r+\bar{r}\}$ be any one of E_1, E_2, E_3, E_4 given in (3.2), and let

$$\alpha_j(t) = x_r(t^\theta)x_{\bar{r}}(t)x_{r+\bar{r}}(t^{\theta+1}),$$
$$\alpha_k(t) = x_{r+\bar{r}}(t^\theta)x_{2r+\bar{r}}(t)$$

be the σ-invariant elements associated with E_i. If $t \neq 0$, then

$$\alpha'_k(t) = \alpha_j(t^{1-2\theta})^{-1}h(\chi)\omega_{\bar{r}}\omega_{2r+\bar{r}}\alpha_j(t^{1-2\theta}),$$

where χ is given by

$$\chi = \chi_{\bar{r},t^{1-2\theta}}\chi_{2r+\bar{r},t^{-1}}.$$

Proof. We have

$$\alpha'_k(t) = x_{-(r+\bar{r})}(t^\theta)\,x_{-(2r+\bar{r})}(t).$$

By (6.2) in [4], we have

$$x_{-(2r+\bar{r})}(t) = x_{2r+\bar{r}}(t^{-1})\,h(\chi_1)\omega_{2r+\bar{r}}x_{2r+\bar{r}}(t^{-1}),$$

where

$$\chi_1 = \chi_{2r+\bar{r},t^{-1}}.$$

Since, by (2.5),

$$\begin{aligned} &x_{-(r+\bar{r})}(t^\theta)x_{2r+\bar{r}}(t^{-1})\,x_{-(r+\bar{r})}(t^\theta) \\ &\quad = x_{2r+\bar{r}}(t^{-1})x_r(t^{\theta-1})\,x_{-\bar{r}}(t^{2\theta-1}) \\ &\quad = x_{2r+\bar{r}}(t^{-1})x_r(t^{\theta-1})x_{\bar{r}}(t^{1-2\theta})h(\chi_2)\omega_{\bar{r}}\,x_{\bar{r}}(t^{1-2\theta}), \end{aligned}$$

where

$$\chi_2 = \chi_{\bar{r},t^{1-2\theta}},$$

we have

$$\begin{aligned}\alpha'_k(t) &= x_{2r+\bar{r}}(t^{-1})\,x_r(t^{\theta-1})\,x_{\bar{r}}(t^{1-2\theta})\,h(\chi_2)\,\omega_{\bar{r}}\,x_{\bar{r}}(t^{1-2\theta})\\ &\quad\cdot x_{-(r+\bar{r})}(t^{\theta})\,h(\chi_1)\,\omega_{2r+\bar{r}}\,x_{2r+\bar{r}}(t^{-1})\\ &= x_{2r+\bar{r}}(t^{-1})\,x_r(t^{\theta-1})\,x_{\bar{r}}(t^{1-2\theta})\,h(\chi_2)\,h(\chi_1\circ w_{\bar{r}})\,\omega_{\bar{r}}\\ &\quad\cdot x_{\bar{r}}(\chi_1(\bar{r})^{-1}t^{1-2\theta})\,x_{-(r+\bar{r})}(\chi_1(r+\bar{r})t^{\theta})\,\omega_{2r+\bar{r}}\,x_{2r+\bar{r}}(t^{-1}).\end{aligned}$$

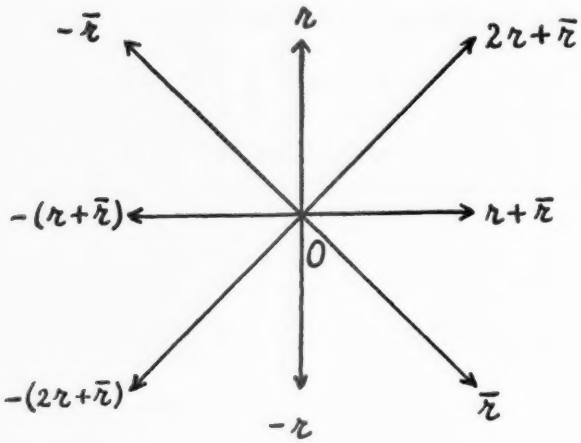

FIGURE 1.

We have

$$\chi_1(\bar{r}) = 1,\ \chi_1(r+\bar{r}) = t^{-1},\ \text{and}\ \chi_1\circ w_{\bar{r}} = \chi_1,$$

since $w_{\bar{r}}(2r+\bar{r}) = 2r+\bar{r}$. Hence

$$\begin{aligned}\alpha'_k(t) &= x_{2r+\bar{r}}(t^{-1})\,x_r(t^{\theta-1})\,x_{\bar{r}}(t^{1-2\theta})\,h(\chi)\,\omega_{\bar{r}}\,\omega_{2r+\bar{r}}\\ &\quad\cdot x_{\bar{r}}(t^{1-2\theta})\,x_r(t^{\theta-1})\,x_{2r+\bar{r}}(t^{-1}),\end{aligned}$$

where χ is as given in (3.10). From this and (2.5) one can derive the desired result easily. Thus (3.10) is proved.

Set $t=1$ in (3.10). Then $h(\chi)=1$. Hence we have $\omega_{\bar{r}}\omega_{2r+\bar{r}} \in G^1$. Hence by setting $E_i = E_1, E_2, E_3, E_4$ we obtain

(3.11) G^1 contain the elements $\omega_{12'}\omega_{12}$, $\omega_{34'}\omega_{34}$, $\omega_{24}\omega_{13}$, $\omega_{24'}\omega_{13'}$.

Now, any one of E_5, E_6, E_7, E_8 is of the form $E_i = \{r, \bar{r}\}$, where $r+\bar{r} \notin \Sigma$, $r-\bar{r} \notin \Sigma$. Hence if $\alpha_j(t) = x_r(t)\,x_{\bar{r}}(t)$ (we assume that $\lambda(r)=1$) are the σ-invariant elements associated with E_i, then

$$\omega_r\omega_{\bar{r}} = \alpha_j(1)\,\alpha'_j(1)\,\alpha_j(1) \in G^1.$$

Hence, by setting $E_i = E_5, E_6, E_7, E_8$, we obtain

(3.12) G^1 contains $\omega_{1234'}\omega_{14}$, $\omega_{123'4}\omega_{14'}$, $\omega_{12'34}\omega_{23}$, $\omega_{12'3'4'}\omega_{23'}$.

It can be easily seen that the elements $w_{12'}w_{12}$, $w_{34'}w_{34}$, $w_{24}w_{13}$, $w_{24'}w_{13'}$, $w_{1234'}w_{14}$, $w_{123'4}w_{14'}$, $w_{12'34}w_{23}$, and $w_{12'3'4'}w_{23'}$ generate the group W^1 defined in (1.10). Hence (3.11) and (3.12) yield

(3.13) For every $w \in W^1$ we can take $\omega(w)$ in G^1 so that $\omega(w)^\sigma = \omega(w)$.

In what follows we shall take $\omega(w)$ in G^1 for all $w \in W^1$, $\omega(1) = 1$.

Now we shall consider the group $\mathfrak{H}^1$ of σ-invariant elements in $\mathfrak{H}$. As in Theorem 4.8 of [4] we have

$$h(\chi)^\sigma = h(\tilde{\chi}),$$

where $\tilde{\chi}$ is defined by

$$\tilde{\chi}(r) = \chi(\psi(r))^\theta = \chi(\bar{r})^{\lambda(r)\theta}.$$

Hence we can deduce easily

(3.14) $h(\chi) \in \mathfrak{H}^1$ if and only if

$$\chi(\xi_2) = \chi(\xi_1)^{2\theta-1},\ \chi(\xi_4) = \chi(\xi_3)^{2\theta-1}.$$

If $E_i = \{r, \bar{r}, r+\bar{r}, 2r+\bar{r}\}$ is any one of E_1, E_2, E_3, E_4, then by (3.11) we have $h(\chi) \in G^1$, where χ is given in (3.10). A simple calculation shows that if $E_i = E_1$, then

$$\chi(\xi_1) = t^{-2\theta}, \qquad \chi(\xi_2) = t^{2\theta-2}, \quad \chi(\xi_3) = \chi(\xi_4) = 1.$$

On the other hand, if $E_i = E_2$, then

$$\chi(\xi_1) = \chi(\xi_2) = 1, \qquad \chi(\xi_3) = t^{-2\theta}, \qquad \chi(\xi_4) = t^{2\theta-2}.$$

Since t is an arbitrary element in K^*, we have by (3.14)

(3.15) $$\mathfrak{H}^1 \subseteq G^1$$

Now we shall prove

(3.16) G^1 is the group of all σ-invariant elements in G. Any element $x \in G^1$ is written uniquely as $x = uh(\chi)\omega(w)u'$, where $u \in \mathfrak{U}^1$, $h(\chi) \in \mathfrak{H}^1$, $w \in W^1$, $u' \in \mathfrak{U}_w \cap \mathfrak{U}^1$.

Proof. By (1.13) of [4], any $x \in G$ can be written uniquely as $x = uh(\chi)\omega(w)u'$, where $u \in \mathfrak{U}$, $h(\chi) \in \mathfrak{H}$, $w \in W$, $u' \in \mathfrak{U}_w$. Then

$$x^\sigma = u^\sigma h(\tilde{\chi})\omega(w)^\sigma u'^\sigma.$$

Proceeding as in the proof of (5.16) of [4], we have $\zeta(\omega(w)^\sigma) = \bar{w}$ and $(\mathfrak{U}_w)^\sigma = \mathfrak{U}_{\bar{w}}$. Hence $\omega(w)^\sigma = h(\chi_1)\omega(\bar{w})$ and $u'^\sigma \in \mathfrak{U}_w$. Therefore, if $x^\sigma = x$,

then by the uniqueness we have $u^\sigma = u$, $\chi = \tilde{\chi}\chi_1$, $\bar{w} = w$, $u'^\sigma = u'$. Hence $\chi_1 = 1$, $\chi = \tilde{\chi}$. Now (3.16) follows from (3.13) and (3.15).

4. The center of $\mathfrak{U}^1$. As a preparation for the proof of the simplicity of G^1, we shall prove here that the center of $\mathfrak{U}^1$ consists of the elements $\alpha_{12}(t)$, $t \in K$. In this section, we do *not* assume that K has more than 2 elements, although we will prove, in the next section, the simplicity of G^1 under that assumption.

Let the groups $\mathfrak{U}^1, \mathfrak{U}^2, \cdots, \mathfrak{U}^{12}$ be as in the preceding section. By (3.8), $\mathfrak{U}^i$ is clearly a normal subgroup of $\mathfrak{U}^1$.

All the expressions for the commutators

$$(\alpha_i(t), \alpha_j(u)) = \alpha_i(t)^{-1}\alpha_j(u)^{-1}\alpha_i(t)\alpha_j(u)$$

used throughout this section can be derived easily from (2.2)-(2.5).

Now, let

$$c = \alpha_1(t_1)\alpha_2(t_2)\cdots\alpha_{12}(t_{12})$$

be an element in the center of $\mathfrak{U}^1$.

We shall first prove $t_1 = t_4 = 0$. We have

$$(\alpha_i(t), \alpha_{10}(u)) = 1 \qquad (i \neq 1, 4) \tag{4.1}$$

Note that (4.1) for $i \geqq 8$ follows from (3.7). Also

$$(\alpha_1(t), \alpha_{10}(u)) = \alpha_{11}(tu). \tag{4.2}$$

$$(\alpha_4(t), \alpha_{10}(u)) = \alpha_{12}(tu). \tag{4.3}$$

Hence $\alpha_{10}(u)^{-1}c\alpha_{10}(u)$ is equal to

$$\alpha_1(t_1)\alpha_{11}(t_1u)\alpha_2(t_2)\alpha_3(t_3)\alpha_4(t_4)\alpha_{12}(t_4u)\alpha_5(t_5)\cdots\alpha_{12}(t_{12}).$$

Equating this to c, and noting that $\alpha_{12}(t)$ is in the center of $\mathfrak{U}^1$, we have $\alpha_{11}(t_1u)\alpha_{12}(t_4u) = 1$. Hence $t_1u = t_4u = 0$ for all $u \in K$. Hence we obtain $t_1 = t_4 = 0$.

We shall prove $t_3 = 0$. We have

$$(\alpha_i(t), \alpha_2(u)) = 1 \qquad (i = 2, 5, 6) \tag{4.4}$$

$$\begin{aligned}(\alpha_3(t), \alpha_2(u)) = {} & \alpha_5(t^{2\theta}u)\alpha_6(tu)\alpha_7(tu^{2\theta})\alpha_8(t^{2\theta+1}u) \\ & \cdot \alpha_9(t^{2\theta+1}u^{2\theta})\alpha_{10}(t^{2\theta+1}u^2)\alpha_{12}(t^{2\theta+2}u^{2\theta+1}).\end{aligned} \tag{4.5}$$

By (4.4), (4.5), above, we can easily compare $c \pmod{\mathfrak{U}^6}$ and $\alpha_2(u)^{-1}c\alpha_2(u) \pmod{\mathfrak{U}^6}$, and obtain $t_3 = 0$.

We shall prove $t_2 = 0$. By (4.4) and (4.5) we have

$$(\alpha_2(t), \alpha_3(u)) \equiv \alpha_5(tu^{2\theta})^{-1} \pmod{\mathfrak{U}^6}.$$

Combining this with

$$(\alpha_5(t), \alpha_3(u)) = \alpha_8(tu) \tag{4.6}$$

we can derive easily $t_2 = 0$.

For the determination of $t_5, t_6, \cdots$, we need

$$(\alpha_6(t), \alpha_3(u)) = \alpha_8(tu^{2\theta})\alpha_9(t^{2\theta}u)\alpha_{12}(t^{2\theta+1}u), \tag{4.7}$$

$$(\alpha_7(t), \alpha_3(u)) = \alpha_9(tu^{2\theta})\alpha_{10}(t^{2\theta}u), \tag{4.8}$$

$$(\alpha_i(t), \alpha_3(u)) = 1 \quad (i = 8, 9, 10, 12), \tag{4.9}$$

$$(\alpha_{11}(t), \alpha_3(u)) = \alpha_{12}(tu). \tag{4.10}$$

By (4.7)-(4.10), above, $\alpha_3(u)^{-1}c\alpha_3(u)$ is equal to

$$\alpha_5(t_5)\alpha_8(t_5u)\alpha_6(t_6)\alpha_8(t_6u^{2\theta})\alpha_9(t_6{}^{2\theta}u)\alpha_{12}(t_6{}^{2\theta+1}u)\alpha_7(t_7)\alpha_9(t_7u^{2\theta})$$
$$\cdot\alpha_{10}(t_7{}^{2\theta}u)\alpha_8(t_8)\alpha_9(t_9)\alpha_{10}(t_{10})\alpha_{11}(t_{11})\alpha_{12}(t_{11}u)\alpha_{12}(t_{12}).$$

Equating this with c and simplifying the relation by

$$\begin{aligned}(\alpha_8(t), \alpha_6(u)) &= (\alpha_9(t), \alpha_8(u)) = (\alpha_9(t), \alpha_7(u)) \\ &= (\alpha_{10}(t), \alpha_8(u)) = (\alpha_{10}(t), \alpha_9(u)) = 1,\end{aligned} \tag{4.11}$$

$$(\alpha_8(t), \alpha_7(u)) = \alpha_{12}(tu) \tag{4.12}$$

we obtain

$$t_5u + t_6u^{2\theta} = 0, \quad t_6{}^{2\theta}u + t_7u^{2\theta} = 0, \quad t_7{}^{2\theta}u = 0,$$
$$(t_5u + t_6u^{2\theta})t_7 + t_6{}^{2\theta+1}u + t_{11}u = 0$$

for all $u \in K$. Hence we have $t_5 = t_6 = t_7 = t_{11} = 0$.

Now, simplifying the relation $\alpha_7(u)^{-1}c\alpha_7(u) = c$ by (4.11), (4.12), and $(\alpha_{10}(t), \alpha_7(u)) = (\alpha_{11}(t), \alpha_7(u)) = 1$, we obtain $t_8 = 0$.

Similarly, simplifying the relation $\alpha_6(u)^{-1}c\alpha_6(u) = c$ by $(\alpha_{10}(t), \alpha_6(u)) = (\alpha_{11}(t), \alpha_6(u)) = 1$ and

$$(\alpha_9(t), \alpha_6(u)) = \alpha_{12}(tu), \tag{4.13}$$

we obtain $t_9 = 0$.

Finally, simplifying the relation $\alpha_4(u)^{-1}c\alpha_4(u) = c$ by (4.3) and $(\alpha_{11}(t), \alpha_4(u)) = 1$, we obtain $t_{10} = 0$. Thus we have proved

THEOREM 4.14. *The center of* $\mathfrak{U}^1$ *consists of the elements* $\alpha_{12}(t)$, $t \in K$.

5. The simplicity of G^1. We shall prove the simplicity of G^1 under the assumption that K contains more than 2 elements. If K contains only 2 elements, then G^1 is of order $2^{12} \cdot 3^3 \cdot 5^2 \cdot 13$ (see Section 6). The author does not know the composition factors of this group.

First we shall prove a lemma which requires the above mentioned assumption.

(5.1) If $w \in W^1$, $w \neq 1$, then there exists an $h \in \mathfrak{H}^1$ such that $\omega(w)h \neq h\omega(w)$.

Proof. Since $\omega(w)h(\chi)\omega(w)^{-1} = h(\chi')$, where χ' is defined by $\chi'(r) = \chi(w^{-1}(r))$, it suffices to derive a contradiction by assuming that $\chi(wr) = \chi(r)$ for all $r \in \mathfrak{Z}$ and all χ such that $h(\chi) \in \mathfrak{H}^1$.

If $w = w(\epsilon, \delta, \infty)$ or $w(\infty, \epsilon, \delta)$ (see (1.10)), then $w \neq 1$ implies that $w(\xi_1) = -\xi_1$ or $w(\xi_1) = \pm \xi_3$. Then $\chi(\xi_1)^2 = 1$ or $\chi(\xi_1) = \chi(\pm \xi_3)$. This is clearly impossible, since, by (3.14), $\chi(\xi_1)$ and $\chi(\xi_3)$ can be regarded as arbitrary elements in K^*, and since K contains more than 2 elements.

If $w = w(\epsilon, \delta, \eta)$, then from $\chi(w\xi_1) = \chi(\xi_1)$ and $\chi(w\xi_3) = \chi(\xi_3)$ one derives easily $\chi(\xi_1 + \xi_2)^\epsilon = \chi(\xi_1)\chi(\xi_3)^\eta$ which is also clearly impossible by the reason stated above. Thus (5.1) is proved.

Now we shall proceed to prove the simplicity of G^1. Let $H \neq \{1\}$ be a normal subgroup of G^1. By (3.16), H contains an element $x = uh\omega(w) \neq 1$ with $u \in \mathfrak{U}^1$, $h \in \mathfrak{H}^1$, and $w \in W^1$. If $w \neq 1$, then, by (5.1), there exists $h_1 \in \mathfrak{H}^1$ such that $\omega(w)h_1 \neq h_1\omega(w)$. We have $1 \neq xh_1x^{-1}h_1^{-1} \in \mathfrak{U}^1\mathfrak{H}^1 \cap H$. Thus we have proved that H contains an element $x = uh \neq 1$, where $u \in \mathfrak{U}^1$, $h \in \mathfrak{H}^1$. Suppose $h = h(\chi) \neq 1$. Then H contains, for any $t \in K^*$,

$$y = x\alpha_{12}(t)x^{-1}\alpha_{12}(t)^{-1} = \alpha_{12}((\chi(\xi_1 + \xi_2) - 1)t)$$

and

$$y' = x\alpha_2(t)x^{-1}\alpha_2(t)^{-1} = \alpha_2((\chi(\xi_3 + \xi_4) - 1)t)y'',$$

where $y'' \in \mathfrak{U}^3$, the group generated by the elements $\alpha_3(t), \cdots, \alpha_{12}(t)$, $t \in K$. Hence if $y = y' = 1$ for all $t \in K^*$, then by (3.4) we would have $\chi(\xi_1 + \xi_2) = \chi(\xi_3 + \xi_4) = 1$, and hence by (3.14), $\chi = 1$, a contradiction. Thus we have proved that H contains an element $x \neq 1$ in $\mathfrak{U}^1$. Suppose x is not in the center of $\mathfrak{U}^1$. Then there exists $y \in \mathfrak{U}^1$ such that the commutator $(x, y) \neq 1$. If $x \in \mathfrak{U}^i$, then, by (3.8), $(x, y) \in \mathfrak{U}^{i+1}$. Hence H must contain an element $\neq 1$ in the center of $\mathfrak{U}^1$. By (4.16), H contains $\alpha_{12}(t)$ with some $t \neq 0$.

Then by considering $h(\chi)\alpha_{12}(t)h(\chi)^{-1}$ we see easily that H contains $\alpha_{12}(t)$ for all $t \in K$.

Since G^1 contains $\omega_0 = \omega(w(-1,-1,\infty))$ by (3.13), H also contains $\alpha'_{12}(t) = \omega_0\alpha_{12}(t)\omega_0^{-1}$ for all $t \in K$. Now, setting $r = \xi_2$ in (3.10), we have $\alpha_k = \alpha_{12}$. Hence by (3.10), H contains $h(\chi)\omega_{12}\omega_{12}$, where χ is defined in (3.10) with $r = \xi_2$. If $t = 1$, then $h(\chi) = 1$. Hence H contains $\omega_{12}\omega_{12}$ and $h(\chi)$ for all $t \in K^*$. We have

$$\chi(\xi_1) = t^{-2\theta}, \quad \chi(\xi_2) = t^{2\theta-2}, \quad \chi(\xi_3) = \chi(\xi_4) = 1. \tag{5.2}$$

Hence, for any $u \in K$ and $t \in K^*$, H contains

$$\begin{aligned}\alpha_5(u)h(\chi)\alpha_5(u)^{-1}h(\chi)^{-1} &= \alpha_5(u)\alpha_5(t^{2-4\theta}u)^{-1} \\ &= \alpha_5(u + t^{2-4\theta}u)\alpha_{12}(u^{2\theta+1}(t^{2\theta-4} + t^{-2\theta-2}))\end{aligned}$$

Therefore H contains $\alpha_5(u + t^{2\theta-4}u)$ for all $u \in K$ and $t \in K^*$. Since $1 + t^{2\theta-4} = 0$ implies $t = 1$, and since K has more than 2 elements, it follows that H contains $\alpha_5(t)$ for all $t \in K$. Thus we have shown that H contains, for all t and u in K, the elements $\alpha_5(t)\alpha_{12}(u)$, which are precisely the σ-invariant elements associated with the set E_1 given in (3.2).

Since $w(\infty,1,1)$, $w(1,1,1)$, $w(1,-1,1)$ map E_1 onto E_2, E_3, E_4, respectively, it follows from (1.6) of [4] and (3.13) that H contains all the σ-invariant elements associated with E_2, E_3, E_4 also.

Now we shall turn to the σ-invariant elements associated with the sets $E_5, \cdots, E_8$. We have already seen in the above that H contains $h(\chi)$ with χ defined in (5.2). Hence H contains, for any $u \in K$ and $t \in K^*$,

$$h(\chi)\alpha_{10}(u)h(\chi)^{-1}\alpha_{10}(u)^{-1} = \alpha_{10}(u + t^{-2\theta}u).$$

Since $1 + t^{-2\theta} = 0$ implies $t = 1$ and since K contains more than 2 elements, it follows from the above that H contains $\alpha_{10}(t)$ for all $t \in K$. Now, $\alpha_{10}(t)$, $t \in K$, are exactly the σ-invariant elements associated with the set E_5. Since $w(1,-1,\infty)$, $w(\infty,1,1)$, $w(-1,1,\infty)$ map E_5 onto E_6, E_7, E_8, respectively, we conclude, as above, that H contains all the σ-invariant elements associated with the sets E_6, E_7, and E_8.

Thus H contains all the σ-invariant elements associated with $E_1, \cdots, E_8$. This implies that H contains $\alpha_1(t), \cdots, \alpha_{12}(t)$, for all $t \in K$, since any $\alpha_i(t)$ is associated with some E_j. Since G^1 contains $\omega_0 = \omega(w(-1,-1,\infty))$, it follows that H contains $\alpha'_i(t) = \omega_0\alpha_i(t)\omega_0^{-1}$ for all $i = 1, \cdots, 12$, and $t \in K$. Therefore we have $H = G^1$. Thus we have proved

THEOREM 5.3. *If K possesses more than 2 elements, then the group G^1 is simple.*

6. The finite groups. Let K be a finite field of $q=2^{2n+1}$ elements, where $n=0,1,2,\cdots$. Then the automorphism $t\to t^\theta$ defined by $t^\theta=t^m$, where $m=2^n$, satisfies $2\theta^2=1$. Hence the group G^1 can be defined over K. We shall compute the order of G^1. By (3.16), G^1 can be partitioned into the 16 subsets:

$$\mathfrak{U}^1\mathfrak{H}^1\omega(w)(\mathfrak{U}_w\cap\mathfrak{U}^1),$$

where w runs over the 16 elements of W^1. The order of $\mathfrak{U}^1$ is, by (3.4), q^{12}, and the order of $\mathfrak{H}^1$ is, by (3.14), $(q-1)^2$. The order of $\mathfrak{U}_w\cap\mathfrak{U}^1$ is, by (3.9), $q^{n(w)}$, where $n(w)$ denotes the number of indices $i=1,\cdots,12$, such that $w(R_i)<0$. Hence the order of G^1 is

$$q^{12}(q-1)^2\sum q^{n(w)}, \tag{6.1}$$

where the sum runs over all $w\in W^1$.

We shall compute the numbers $n(w)$, $w\in W^1$. Note first that if $\alpha_i(t)$ and $\alpha_j(t)$ are associated with the same set E_k, then $w(E_i)<0$ if and only if $w(E_j)<0$. In fact, it can be easily seen that, if $w(r)<0$ for some $r\in E_k$, then $w(r)<0$ for all $r\in E_k$. For $w\in W^1$, define $\nu(w;E_k)=1$ or -1 according as $w(r)>0$ for all $r\in E_k$ or not. Then we have

$$n(w)=\sum_{1\leqq k\leqq 4}(1-\nu(w;E_k))+\tfrac{1}{2}\sum_{5\leqq k\leqq 8}(1-\nu(w;E_k)). \tag{6.2}$$

The numbers $\nu(w;E_k)$ can be computed easily by (1.10) and (3.2). We have

The table of $\nu(w;E_k)$

	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
$w(\epsilon,\delta,\infty)$	ϵ	δ	ϵ	ϵ	ϵ	ϵ	ϵ	ϵ
$w(\infty,\epsilon,\delta)$	ϵ	δ	δ	$-\delta$	δ	$-\delta$	ϵ	$-\epsilon$
$w(\epsilon,\delta,1)$	ϵ	ϵ	ϵ	δ	ϵ	ϵ	ϵ	$-\epsilon$
$w(\epsilon,\delta,-1)$	ϵ	$-\epsilon$	δ	ϵ	ϵ	ϵ	$-\epsilon$	ϵ

Hence by (6.2) we have

$$\begin{aligned}
n(w(\epsilon,\delta,\infty)) &= 6-5\epsilon-\delta = 0,2,10,12,\\
n(w(\infty,\epsilon,\delta)) &= 6-\epsilon-\delta = 4,6,6,8,\\
n(w(\epsilon,\delta,1)) &= 6-4\epsilon-\delta = 1,3,9,11,\\
n(w(\epsilon,\delta,-1)) &= 6-2\epsilon-\delta = 3,5,7,9.
\end{aligned}$$

Therefore $\sum q^{n(w)}$ is equal to

$$1+q+q^2+2q^3+q^4+q^5+2q^6+q^7+q^8+2q^9+q^{10}+q^{11}+q^{12}$$
$$=(q+1)(q^2+1)(q^3+1)(q^6+1).$$

Hence by (6.1) we have

THEOREM 6.3. *If K is a finite field of $q=2^{2n+1}$ elements, where $n=0,1,2,\cdots$, then the order of G^1 is given by*

$$q^{12}(q-1)(q^3+1)(q^4-1)(q^6+1).$$

7. Automorphisms of G^1. If $t\to t'$ is an automorphism of the field K which commutes with θ, then there exists an automorphism ρ of G^1 such that $\alpha_i(t)\to\alpha_i(t')$, $\alpha'_i(t)\to\alpha'_i(t')$ for all $t\in K$ and i. We shall call ρ the automorphism of G^1 induced by the automorphism $t\to t'$ of K. We shall prove

THEOREM 7.1. *If K is a finite*[8] *field possessing more than 2 elements, then every automorphism ρ of the group G^1 can be written uniquely in the form $\rho=\rho_1\rho_2$, where ρ_1 is an automorphism induced by an automorphism of K, and where ρ_2 is an inner automorphism of G^1.*

Proof. Arguing as in the proof of Theorem 9.1 of [4], we see that we can assume (7.2)-(7.5), below:

$$(\mathfrak{U}^1)^\rho=\mathfrak{U}^1, \tag{7.2}$$

$$\alpha_{12}(t)^\rho=\alpha_{12}(t) \quad \text{for all } t\in K, \tag{7.3}$$

$$h(\chi)^\rho=h(\chi) \quad \text{for all } h(\chi)\in\mathfrak{H}^1, \tag{7.4}$$

$$\omega_0{}^\rho=h(\chi_0)\omega_0 \quad \text{where } \omega_0=\omega(w(-1,-1,\infty)). \tag{7.5}$$

Define the roots $r_1,r_2,\cdots,r_{12}$ by setting $r_i=\bar{r}$ if $\alpha_i(t)=x_r(t^\theta)x_{\bar{r}}(t)$ or if $\alpha_i(t)=x_r(t^\theta)x_{\bar{r}}(t)x_{r+\bar{r}}(t^{\theta+1})$. Then the roots $r_1,\cdots,r_{12}$ are respectively,

$$34',34,23',24',12',24,23,13',14',14,13,12,$$

where ij and ij' denote $\xi_i+\xi_j$ and $\xi_i-\xi_j$, respectively. From the definition of r_i we have

$$h(\chi)\alpha_i(t)h(\chi)^{-1}=\alpha_i(\chi(r_i)t) \tag{7.6}$$

[8] J. Tits remarks that the finiteness assumption seems to be dispensed with if one uses the geometric interpretation of the group G^1 given in [10].

for all $h(\chi) \in \mathfrak{H}^1$ and $t \in K$. Now fix an i, and set

$$\alpha_i(t)^\rho = \alpha_1(t^{(1)})\alpha_2(t^{(2)}) \cdots \alpha_{12}(t^{(12)}).$$

Then by (7.4) and (7.6) we have

$$(\chi(r_i)t)^{(j)} = \chi(r_j)t^{(j)} \tag{7.7}$$

Hence if for the root r_j, there exists $h(\chi) \in \mathfrak{H}^1$ such that $\chi(r_i) = 1$ and $\chi(r_j) \neq 1$, then we have $t^{(j)} = 0$ for all $t \in K$. Using this, (3.13), and the assumption that K has more than 2 elements, we can deduce (7.8)-(7.10), below:

(7.8) If $i \neq 1, 2, 5$, then $\alpha_i(t)^\rho = \alpha_i(c_i t)$, where c_i is a constant in K.

(7.9) Both $\alpha_1(t)^\rho$ and $\alpha_2(t)^\rho$ are of the form $\alpha_1(t_1)\alpha_2(t_2)$.

(7.10) $\alpha_5(t)^\rho$ is of the form $\alpha_5(t_1)\alpha_{12}(t_2)$.

It can be verified easily by (2.1) that the center of the group generated by the elements $\alpha_1(t)$, $\alpha_2(t)$, $t \in K$, consists of the elements $\alpha_2(t)$, $t \in K$. From this, (7.7), and (7.8) that there exists a constant c_2 in K such that $\alpha_2(t)^\rho = \alpha_2(c_2 t)$ for all $t \in K$. Now take an element $h(\chi) \in \mathfrak{H}^1$ such that $\chi(\xi_3 + \xi_4) = c_2$, and define an inner automorphism ρ' of G^1 by $x^{\rho'} = h(\chi)^{-1} x h(\chi)$. Then $\alpha_2(t)^{\rho\rho'} = \alpha_2(t)$ for all $t \in K$, and all the statements (7.2)-(7.10) remain valid when ρ is replaced by $\rho\rho'$. Hence, for the proof of Theorem 7.1, we can furthermore assume

$$\alpha_2(t)^\rho = \alpha_2(t) \tag{7.11}$$

By (7.7) and (7.9) we can set

$$\alpha_1(t)^\rho = \alpha_1(c_1 t)\alpha_2(c' t^{2\theta+1}),$$

where c_1 and c' are constants in K. Combining this with the identity

$$\alpha_1(t)\alpha_1(u) = \alpha_1(t+u)\alpha_2(tu^{2\theta})$$

we obtain

$$c_1{}^{2\theta+1} t u^{2\theta} + c' t^{2\theta+1} + c' u^{2\theta+1} = t u^{2\theta} + c'(t+u)^{2\theta+1}$$

for all $t, u \in K$. By setting $t = u = 1$ we have $c_1{}^{2\theta+1} = 1$, $c_1 = 1$. Then from the above it follows that $c'(t^{2\theta} u + t u^{2\theta}) = 0$ for all $t, u \in K$. From this and the assumption that K has more than 2 elements we can derive easily $c' = 0$. Thus $\alpha_1(t)^\rho = \alpha_1(t)$ for all $t \in K$. Similarly, we can prove $\alpha_5(t)^\rho = \alpha_5(t)$ for all $t \in K$. Thus

$$\alpha_1(t)^\rho = \alpha_1(t), \qquad \alpha_5(t)^\rho = \alpha_5(t). \tag{7.12}$$

Now, applying ρ to the identity (4.7) and using (7.8), (7.11), and (7.12), we obtain $c_3 = c_6 = c_7 = c_8 = c_9 = c_{10} = 1$. Similarly from (4.4) we obtain $c_4 = 1$. Thus we have proved $c_i = 1$ for all i. Thus

$$\alpha_i(t)^\rho = \alpha_i(t) \qquad (i = 1, 2, \cdots, 12) \tag{7.13}$$

By setting $r = \xi_2, \xi_4$ in (3.10) we obtain, respectively,

$$\alpha'_{12}(t) = \alpha_5(t^{1-2\theta})^{-1} h' \omega' \alpha_5(t^{1-2\theta}),$$
$$\alpha'_2(u) = \alpha_1(u^{1-2\theta})^{-1} h'' \omega'' \alpha_1(u^{1-2\theta}),$$

where: $\omega' = \omega(w(-1, 1, \infty))$; $\omega'' = \omega(w(1, -1, \infty))$; h' is an element in $\mathfrak{H}^1$ which commutes with ω'' and the elements $\alpha_1(t)$, $t \in K$; h'' is an element in $\mathfrak{H}^1$ which commutes with ω' and the elements $\alpha_5(t)$, $t \in K$. Since $w(-1, 1, \infty) w(1, -1, \infty) = w(-1, -1, \infty)$, we can assume $\omega'\omega'' = \omega_0 = \omega(w(-1, -1, \infty))$. Then, since $(\alpha_1(t), \alpha_5(u)) = 1$, we have from the above,

$$\alpha'_2(u)\alpha'_{12}(t) = (\alpha_1(u^{1-2\theta})\alpha_5(t^{1-2\theta}))^{-1} h \omega_0 \alpha_1(u^{1-2\theta})\alpha_5(t^{1-2\theta})$$

with $h = h'h'' \in \mathfrak{H}^1$. We have from (7.5)

$$\begin{aligned}(\alpha'_2(u)\alpha'_{12}(t))^\rho &= (\omega_0 \alpha_2(u)\alpha_{12}(t)\omega_0^{-1})^\rho \\ &= \alpha'_2(\chi_0(\xi_3 + \xi_4)^{-1}u)\alpha'_{12}(\chi_0(\xi_1 + \xi_2)^{-1}t) \\ &= a^{-1} h h(\chi_0)\omega_0 a \\ &= \alpha'_2(u)\alpha'_{12}(t) a^{-1}\omega_0^{-1} h(\chi_0)\omega_0 a\end{aligned}$$

where $a = \alpha_1(u^{1-2\theta})\alpha_5(t^{1-2\theta})$. From this and the relations

$$\omega_0^{-1} h(\chi_0)\omega_0 = h(\chi_0)^{-1}, \qquad \mathfrak{U}^1\mathfrak{H}^1 \cap \mathfrak{B}^1 = \{1\},$$

we can deduce easily $h(\chi_0) = 1$. Hence (7.5) yields

$$\omega_0^\rho = \omega_0 \tag{7.14}$$

Since the group G^1 is generated by ω_0 and $\alpha_1(t), \cdots, \alpha_{12}(t)$, $t \in K$, we can conclude from (7.13) and (7.14) that $\rho = 1$. Thus Theorem 7.1 is proved except for the uniqueness part.

The uniqueness part, however, can be proved just as the corresponding part of Theorem 9.1 of [4], if one uses the fact that ω_0 belongs to the group generated by the elements $\alpha_1(1)$, $\alpha_2(1)$, $\alpha_5(1)$, $\alpha_{12}(1)$, $\alpha'_1(1)$, $\alpha'_2(1)$, $\alpha'_5(1)$, $\alpha'_{12}(1)$. Thus the proof of Theorem 7.1 is complete.

UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, B. C.

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THE STRUCTURE OF FROBENIUS ALGEBRAIC GROUPS.*

By David Hertzig.[1]

1. Introduction. A finite group G is called a Frobenius group if it has a proper subgroup H which is "disjoint" from all of its conjugates (i.e. $xHx^{-1} \cap H \neq \{e\}$ implies $x \in H$). The set $M = \{G - \bigcup_{x \in G} xHx^{-1}\} \cup \{e\}$ is then easily seen to be a normal subgroup, called the regular subgroup of G. The structure of M has been greatly studied and a long outstanding conjecture that M is necessarily nilpotent has recently been proved by J. Thompson [4].

In this paper we study the structure of Frobenius algebraic groups. An algebraic group G is called a *Frobenius algebraic group* if it has a proper algebraic subgroup H such that for all $x \in G$, $xHx^{-1} \cap H \neq \{e\}$ implies $x \in H$. Denote $M = \{G - \bigcup_{x \in G} xHx^{-1}\} \cup \{e\}$. This notation will be kept throughout the paper.

The simple counting techniques of the finite theory are no longer available and consequently the nature of the study of Frobenius algebraic groups is necessarily quite different from that of the finite theory. For example, the fact that M is a subgroup does not come out until quite late (Theorem 4). In the finite case this is proved by Frobenius using the theory of characters. For properties of finite Frobenius groups see [3].

The theory of algebraic groups is assumed. The reader is referred to [1] and [2] for the standard theorems and terminology. In particular, if G is an algebraic group, G_0 denotes the connected component of the identity element e; G_0 is a normal algebraic subgroup of finite index in G. A Borel subgroup of G is a maximal connected solvable algebraic subgroup. A torus of G is a connected diagonalizable subgroup. The set of semisimple (i.e. diagonalizable) elements of G is denoted G_s; the set of unipotent elements of G is denoted G_u. A unipotent group, i.e., one for which $G = G_u$, is nilpotent. If a Borel subgroup of G is nilpotent then G is nilpotent; in particular if a maximal torus of G is reduced to $\{e\}$ G is nilpotent. In general p will denote the characteristic of the ground field.

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2. Basic properties of Frobenius groups. For $g \in G$ we define the rational map $\sigma_g: G \to G$ by $\sigma_g(x) = gxg^{-1}x^{-1}$. If N is a subgroup of G normalized by g then clearly $\sigma_g(N) \subset N$, and if N is a normal subgroup of G, σ_g induces a map (still denoted σ_g) $\sigma_g: G/N \to G/N$.

LEMMA 1. *Let G be a Frobenius algebraic group and $N \supset N_1$ algebraic subgroups of G with N_1 normal in N and such that H normalizes N and N_1. Then*

a) *for all $h \in H$, $h \neq e$, σ_h is injective on N if and only if $N \cap H = \{e\}$;*

b) *if σ_h is injective on N and surjective on N_1 then σ_h is injective on N/N_1;*

c) *if σ_h is surjective on N_1 and on N/N_1 then σ_h is surjective on N.*

Proof. a) Let $n \in N \cap H$; then $\sigma_n(n) = e = \sigma_n(e)$, hence $n = e$. Conversely $\sigma_h(n) = \sigma_h(n')$ implies $nhn^{-1} = n'hn'^{-1}$ and so $n^{-1}n'Hn'^{-1}n \cap H \neq \{e\}$. Therefore $n^{-1}n' \in H \cap N = \{e\}$ and $n = n'$.

b) Suppose $\sigma_h(n) = \sigma_h(n')n_1$ with $n, n' \in N$, $n_1 \in N_1$. Then $n'^{-1}n_1n' \in N_1$ and so $n'^{-1}n_1n' = \sigma_h(n'_1)$ for some $n'_1 \in N_1$. Therefore $\sigma_h(n) = \sigma_h(n')n_1 = \sigma_h(n')n'\sigma_h(n'_1)n'^{-1} = \sigma_h(n'n'_1)$. By hypothesis σ_h is injective on N, so $n = n'n'_1$ and σ_h is injective on N/N_1.

c) Let $n \in N$; then σ_h surjective on N/N_1 implies $n = \sigma_h(n')n_1$ with $n' \in N$, $n_1 \in N_1$. As in the proof of (b) there exists $n'_1 \in N_1$ such that $n = \sigma_h(n'n'_1)$. q. e. d.

LEMMA 2. *Let G be a Frobenius algebraic group and N a solvable algebraic subgroup of G normalized by H. For $h \in H$, if σ_h is injective on N then σ_h is surjective on N.*

Proof. First suppose N connected. If N is commutative σ_h is a homomorphism of N into N and since σ_h is injective $\sigma_h(N)$ and N have the same dimension, hence $\sigma_h(N) = N$. We proceed by induction on the dimension of N. Let N_1 be the commutator subgroup of N; since N is solvable N_1 is of strictly smaller dimension and by the induction hypothesis σ_h is surjective on N_1. By Lemma 1b σ_h is injective on N/N_1, which is commutative, hence σ_h is surjective on N/N_1. Therefore by Lemma 1c σ_h is surjective on N.

In the general case, by what we have already proved σ_h is surjective on N_0 so by Lemma 1b σ_h is injective on N/N_0, which is finite. Therefore σ_h is surjective on N/N_0 and applying Lemma 1c we have σ_h surjective on N.

q. e. d.

LEMMA 3. *Let G be a Frobenius algebraic group and N a solvable normal algebraic subgroup of G such that $N \cap H = \{e\}$. Let ρ be a rational representation of G with kernel N, $\bar{G} = \rho(G)$, $\bar{H} = \rho(H)$. Then $\bar{G}$ is a Frobenius algebraic group.*

Proof. Let $\bar{x} \in \bar{G}$ such that $\bar{x}\bar{H}\bar{x}^{-1} \cap \bar{H} \neq \{\bar{e}\}$. Then if $\bar{x} = \rho(x)$ there exist, h, $h_1 \in H$, $h \neq e$, $h_1 \neq e$, and $n \in N$ such that $xh_1x^{-1} = h^{-1}n$. By Lemma 1a σ_h is injective on N hence by Lemma 2 σ_h is surjective on N; let $n = \sigma_h(n_1)$. Then $xh_1x^{-1} = h^{-1}n = h^{-1}\sigma_h(n_1) = n_1h^{-1}n_1^{-1}$ and so $n_1^{-1}xHx^{-1}n_1 \cap H \neq \{e\}$; therefore $n_1^{-1}x \in H$ and $\bar{x} = \rho(n_1^{-1}x) \in \bar{H}$. q. e. d.

PROPOSITION 1. *Let G be a Frobenius algebraic group. Then $H \cap G_0 = H_0$ and if H is not finite, $H_0 = T$ is a maximal torus of G, G_0 is solvable, $M = G_{0u}$ and $G = H \cdot M$.*

Proof. Suppose H is finite and let $K = H \cap G_0$; if $s \in K_s$, $s \neq e$, then s belongs to a maximal torus T of G_0. If $x \in G_0$ centralizes T, $xsx^{-1} = s \in H \cap xHx^{-1}$ and $x \in K$ so that $T \subset K$ and $T = \{e\}$ which is impossible. Therefore $K = K_u$; let $u \in K$, $u \neq e$. Then $u \in B_u$ for some Borel subgroup B of G and since B_u is unipotent it is nilpotent and consequently has a positive dimensional center Z. If $x \in Z$, $xux^{-1} = u \in H \cap xHx^{-1}$, hence $x \in K$, therefore $Z \subset K$ which is a contradiction. Therefore $K = H \cap G_0 = \{e\} = H_0$.

We suppose then H is not finite, i. e., $H_0 \neq \{e\}$. If $x \in G$ normalizes H_0 then $x \in H$ and so H_0 is equal to its connected normalizer in G. If H_0 is unipotent it is nilpotent, hence is a Cartan subgroup of G_0 and $G_0 = H_0$ so that G normalizes H_0, hence $G = H$ contrary to assumption that H is a proper subgroup of G. Therefore a maximal torus S of H_0 is $\neq \{e\}$ and since S is contained in a maximal torus T of G_0 and T centralizes S, $T \subset H$ and $T = S$.

Let B_1 be a Borel subgroup of H_0 containing T and B a Borel subgroup of G_0 containing B_1. Then $B \cap H_0$ is a solvable subgroup of H_0 containing B_1, which is maximal solvable, hence $B_1 = B \cap H_0$. Similarly, since $(B \cap H)_0 \subset H_0$, $B_1 = B \cap H_0 = (B \cap H)_0$. Therefore $B_{1u} = B_u \cap H_0 = (B_u \cap H)_0$.

Let $x \in B_u$ normalize B_{1u}; then if $B_{1u} \neq \{e\}$, $xHx^{-1} \cap H \neq \{e\}$ and $x \in H$, so that B_{1u} is equal to its connected normalizer in B_u. Since B_u is a connected nilpotent group, it follows that $B_u = B_{1u}$ and since $B = T \cdot B_u$, $B = B_1 \subset H$. Let $g \in G_0$; then gBg^{-1} is a Borel subgroup of G_0 and since any two Borel subgroups contain a common maximal torus, $gHg^{-1} \cap H \neq \{e\}$ and $g \in H$, i. e. $G_0 \subset H$. Hence $G_0 = H_0$ and $G = H$ contrary to hypothesis.

Therefore $B_{1u} = \{e\}$ and $B_1 = T$ so that $H_0 = T$. Let $K = H \cap G_0$ and

suppose $u \in K_u$, $u \neq e$. Then $u \in B_u$ for some Borel subgroup B of G_0 and since $B_u \neq \{e\}$ the connected center Z of $B_u \neq \{e\}$. Let $x \in Z$; since $xux^{-1} = u \in H \cap xHx^{-1}$, $x \in H$, i.e. $Z \subset H$, hence $Z \subset H_0 = T$ which is impossible since Z is unipotent. Therefore $K_u = \{e\}$ and $K = K_s$. Let $s \in K_s$, $s \neq e$; then s is in some maximal torus gTg^{-1} of G_0 and so $s \in H \cap gHg^{-1}$. Therefore $g \in H$ and $gTg^{-1} = T$, i.e. $s \in H_0$ and $K = H \cap G_0 = H_0$. Moreover $N_G(H_0) \subset H$ and so $N_{G_0}(H_0) \subset H \cap G_0 = H_0$, hence $N_{G_0}(T) = T$ and the Weyl group of G_0 is trivial, i.e. G_0 is solvable.

Let $x \in G$; xTx^{-1} is a maximal torus of G_0, hence $xTx^{-1} = yTy^{-1}$ with $y \in G_{0u}$ and $y^{-1}x \in H$ so that $G = G_{0u}H$. Since $G_{0u} \cap H \subset G_0 \cap H = H = T$, $G_{0u} \cap H = \{e\}$ and $G = G_{0u} \cdot H$.

By Lemma 1a, for all $h \in H$, $h \neq e$, σ_h is injective on G_{0u} and so by Lemma 2, σ_h is surjective on G_{0u}. Let $g \in M$, $g \neq e$, $g = hu$, $u \in G_{0u}$, $h \in H$. If $h \neq e$, $u = \sigma_{h^{-1}}(g_0)$ and $g = g_0 h g_0^{-1}$ which is impossible. Therefore $h = e$ and $M \subset G_{0u}$. Conversely, since $G_{0u} \cap H = \{e\}$, $G_{0u} \cap \bigcup_{x \in G} xHx^{-1} = \{e\}$ and $G_{0u} \subset M$, hence $G_{0u} = M$. q. e. d.

Proposition 1 gives the desired structure theorem for Frobenius algebraic groups in the case H not finite. For the remainder of this paper we adopt the convention: an algebraic group G is a *Frobenius algebraic group* if it has a proper finite algebraic subgroup H such that for all $x \in G$, $xHx^{-1} \cap H \neq \{e\}$ implies $x \in H$. Proposition 1 then asserts $H \cap G_0 = \{e\}$.

COROLLARY. *If G is a Frobenius algebraic group then $G_0 \subset M$.*

Proof. By Proposition 1 $G_0 \cap H = \{e\}$ and since G_0 is normal in G, $G_0 \cap \bigcup xHx^{-1} = \{e\}$. Therefore $G_0 \subset \{G - \bigcup_{x \in G} xHx^{-1}\} \cup \{e\} = M$. q. e. d.

3. A first structure theorem. The corollary to Proposition 1 asserts $G_0 \subset M$ hence if M is a subgroup of G (as is proved later in Theorem 4) $G_0 = M_0$. Therefore a first approach to the study of the structure of M is through the structure of G_0. In this section we consider the special case where G_0 is assumed to be solvable.

THEOREM 1. *Let G be a Frobenius algebraic group and suppose G_0 is solvable. Then G_0 is nilpotent. Moreover, if* $\dim G > 0$ *either G_0 is a torus of dimension $d \equiv 0 \pmod{p-1}$, in which case H is a p-group, or G_0 is unipotent, in which case the order of H is prime to p.*

Proof. Let l be a prime dividing the order of H and let h be an element of H of order l. If $\dim G_u > 0$ G_u contains a characteristic subgroup A isomorphic to G_a (the one-dimensional additive group). Since $hAh^{-1} = A$

and no element of A centralizes h the map $\mathrm{ad}\,h: A \to A$ defined by $\mathrm{ad}\,h(a) = hah^{-1}$ induces a fixed-point-free automorphism Φ of G_a. Any automorphism of G_a is of the form $x \to \lambda x^{p^\nu}$ and since $\Phi^l = 1$ we must have $\Phi(x) = \lambda x$ where $\lambda^l = 1$ and $\lambda \neq 1$, hence $l \neq p$. Therefore if $\dim G_u > 0$ the order of H is prime to p.

Now let T_0 be a maximal torus of G_0; then hT_0h^{-1} is also a maximal torus of G_0, hence by conjugacy of maximal tori $hT_0h^{-1} = xT_0x^{-1}$ for some $x \in G_0$. By Proposition 1 and Lemma 1a, σ_h is injective on G_0 and by Lemma 2, σ_h is surjective on G_0 so we have $x = \sigma_h(g)$, $g \in G_0$. Therefore $hT_0h^{-1} = \sigma_h(g)T_0\sigma_h(g)^{-1} = hgh^{-1}g^{-1}T_0ghg^{-1}h^{-1}$, i.e. if $T = g^{-1}T_0g$, $hTh^{-1} = T$. If $d = \dim T$ and n is an integer prime to p the subgroup T_n of elements of T of order n contains exactly n^d elements. Clearly $hT_nh^{-1} = T_n$ and no element of T centralizes H. It follows that $n^d \equiv 1 \pmod{l}$. Therefore if $l \neq p$, in particular $l^d \equiv 1 \pmod{l}$ and so $d = 0$, $T = \{e\}$ and G_0 is unipotent. If H is a p-group, G_0 is a torus and $n^d \equiv 1 \pmod{p}$ for all n prime to p implies $d \equiv 0 \pmod{p-1}$. q. e. d.

4. Fixed-point free automorphisms. An automorphism Φ of a group G is said to be fixed-point-free if the only element of G left fixed by Φ is the identity element. If G is a Frobenius algebraic group and $h \in H$, $h \neq e$, then (assuming M to be a subgroup) $\mathrm{ad}\,h: x \to hxh^{-1}$ induces a fixed-point-free automorphism of M. Conversely, if an algebraic group M admits a rational fixed-point-free automorphism Φ of prime order then, if H denotes the group generated by Φ, the split extension of H by M is a Frobenius algebraic group. The following statements are translations of the corresponding results on Frobenius groups into the language of automorphisms. The proofs of Lemmas 1′, 2′ and 3′ are identical to the proofs of the corresponding results on Frobenius groups; it is clear that here we need no restriction on the order of Φ. If Φ is an automorphism of G, we denote $\bar{\Phi}$ the map of G into G defined by $\bar{\Phi}(x) = \Phi(x)x^{-1}$.

LEMMA 1′. *Let G be an algebraic group, Φ a rational automorphism of G and $N \supset N_1$ algebraic subgroups of G, stable under Φ, with N_1 normal in N. Then*

a) *$\bar{\Phi}$ is injective on N if and only if Φ is fixed-point-free on N;*

b) *if $\bar{\Phi}$ is injective on N and surjective on N_1 then $\bar{\Phi}$ is injective on N/N_1, i.e. if Φ acts fixed-point freely on N and $\bar{\Phi}$ is surjective on N_1 then Φ induces a fixed-point-free automorphism of N/N_1;*

c) *if $\bar{\Phi}$ is surjective on N_1 and on N/N_1 then Φ is surjective on N.*

LEMMA 2′. *Let Φ be an automorphism of the algebraic group G and N a solvable algebraic subgroup of G stable under Φ and such that Φ acts fixed-point freely on N. Then $\bar{\Phi}$ is surjective on N.*

LEMMA 3′. *Let G be an algebraic group admitting a fixed-point-free rational automorphism Φ and N a solvable normal algebraic subgroup of G stable under Φ. Then Φ induces a fixed-point-free automorphism of G/N.*

LEMMA 4. *Let G be a connected algebraic group admitting a fixed-point-free rational automorphism of finite period n. Then for all $x \in G$,*

$$\Phi^{n-1}(x)\Phi^{n-2}(x) \cdot \cdot \cdot \Phi(x)x = e.$$

Proof. For $x, y \in G$ we define $y \circ x = \Phi(y)xy^{-1}$. Then $e \circ x = x$ and $z \circ (y \circ x) = (zy) \circ x$ so that G acts on itself as a transformation space. The isotropy subgroup of e consists precisely of the fixed-points of Φ, hence is reduced to $\{e\}$. Therefore the orbit U of e is an open subset of G (in fact, U is a principal homogeneous space for G). If $g \in U$ we may write $g = \Phi(y)y^{-1}$, hence $\Phi^{n-1}(g)\Phi^{n-2}(g) \cdot \cdot \cdot \Phi(g)g = e$ for all $g \in U$. Since U is dense in G the assertion of the lemma follows immediately. q. e. d.

LEMMA 5. *Let Φ be a fixed-point-free rational automorphism of a torus T. Then if the order of Φ is finite, it is a power of p and if s is prime to p, Φ^s is also fixed-point-free.*

Proof. Suppose $\Phi^n = 1$ and let l be a prime different from p dividing n, say $n = lm$. Let $F = \{x \in T, \Phi^l(x) = x\}$; then F_0 is a torus admitting a fixed-point-free automorphism of period $l \neq p$ and so by Theorem 1 F is finite of order $\equiv 1 \pmod{l}$. Let $T_1 = T/F$ and $F_1 = \{\bar{x} \in T_1, \Phi^m(\bar{x}) = \bar{x}\}$. If $\bar{x} \in T_1$ is such that $\bar{x}^{-1}\Phi^m(\bar{x}) \in F_1$ then $\bar{x}^{-1}\Phi^m(\bar{x}) = \Phi^m(\bar{x})^{-1}\Phi^{2m}(\bar{x}) = \cdots = \Phi^{(l-1)m}(\bar{x})^{-1}\Phi^{lm}(\bar{x})$ and so $[\bar{x}^{-1}\Phi^m(\bar{x})]^l = \bar{e}$, i. e. $[x^{-1}\Phi^m(x)]^l \in F$ and since the order of F is $\equiv 1 \pmod{l}$, $x^{-1}\Phi^m(x) \in F$, hence $\bar{x} \in F_1$. Therefore Φ^m acts fixed-point freely on the torus T_1/F_1 so by Theorem 1 $T_1 = F_1$, i. e. $\Phi^m = 1$ on T_1. Therefore the homomorphism $\rho: x \to x^{-1}\Phi^m(x)$ maps T into F and the image of ρ has dimension 0. Therefore $T =$ kernel of ρ, that is to say $\Phi^m = 1$ on T. Proceeding in this manner we finally have $\Phi^{p^r} = 1$ on T for some power r of p.

If $(s, p) = 1$ then $as + bp^r = 1$ and $\Phi = \Phi^{as}$. Therefore if $x = \Phi^s(x)$, we have $x = \Phi^{as}(x) = \Phi(x)$ so $x = e$. q. e. d.

LEMMA 6. *Let Φ be a fixed-point-free rational automorphism of a unipotent group G. Then Φ^{p^r} also is fixed-point-free.*

Proof. Let $F = \{x \in G, \Phi^{p^r}(x) = x\}$. Since G is unipotent, hence nil-

potent, so is the subgroup F and if $F \neq \{e\}$ then there exists an element $x \neq e$ in the center of F. The group H generated by $x, \Phi(x), \cdots, \Phi^{p^r-1}(x)$ is a finitely generated commutative group, hence a finite p-group. But H admits the fixed-point-free automorphism Φ of period a power of p, hence must have order $\equiv 1 \pmod{p}$, which is impossible. Therefore $F = \{e\}$ and Φ^{p^r} is fixed-point-free. q. e. d.

THEOREM 1′. *Let G be a connected solvable algebraic group admitting a fixed-point-free rational automorphism Φ of order n. Then if n is prime to p, G is unipotent and if n is a power of p, G is a torus. In particular, if n is a power of a prime G is nilpotent. Also, if the restriction of Φ to G_u has order prime to p, G is nilpotent.*

Proof. Let T_1 be a maximal torus of G; then $\Phi(T_1) = xT_1x^{-1}$ and writing $x = \Phi(y)y^{-1}$ (Lemma 2′) the torus $T = y^{-1}T_1y$ is stable under Φ. Since $G = T \cdot G_u$ the first assertion of the theorem is an immediate consequence of Lemmas 5 and 6.

Now suppose the restriction of Φ to G_u has order s prime to p. For $t \in T$, $u \in G_u$, $\Phi^s(tut^{-1}) = \Phi^s(t)\Phi^s(u)\Phi^s(t)^{-1}$ and so $tut^{-1} = \Phi^s(t)u\Phi^s(t)^{-1}$, i.e. $t^{-1}\Phi^s(t)$ centralizes G_u for all $t \in T$. By Lemma 5, Φ^s is fixed-point-free on T and by Lemma 2′ every element of T is of the form $t^{-1}\Phi^s(t)$. Therefore T is in the center of G and $G = T \times G_u$ is nilpotent. q. e. d.

We have seen in the proof of Theorem 1′ that there is a maximal torus T of G stable under Φ and by Lemma 5 the restriction of Φ to T has order a power of p. If m is any integer prime to p the subgroup T_m of elements of T of order m has order m^d where $d = \dim T = \text{rank of } G$. Since T_m admits Φ fixed-point freely $m^d \equiv 1 \pmod{p}$. Therefore we must have $d \equiv 0 \pmod{p-1}$.

COROLLARY. *If the algebraic group G admits a fixed-point-free rational automorphism of finite order then G is solvable of rank $d \equiv 0 \pmod{p-1}$.*

PROPOSITION 2. *Let G be an algebraic group admitting a fixed-point-free rational automorphism Φ of period 2. Then G is commutative and Φ is inversion. Moreover, if $p = 2$, G_0 is a torus and if $p \neq 2$, G_0 is unipotent.*

Proof. By Lemma 4 for all $x \in G_0$, $\Phi(x)x = e$, hence $\Phi(x) = x^{-1}$ and G_0 is commutative. Applying Lemmas 2′ and 1′ we have $\bar{\Phi}$ is injective on G/G_0, hence surjective on G/G_0 since G/G_0 is finite. Then by Lemma 1′c, $\bar{\Phi}$ is surjective on G and therefore for all $x \in G$, $\Phi(x)x = e$, i.e. Φ is inversion and G is commutative.

Since Φ acts fixed-point freely, G has no elements of order 2 and so if $p = 2$, G_0 is a torus and if $p \neq 2$, G_0 is unipotent. q. e. d.

LEMMA 7. *Let G be a connected algebraic group admitting a fixed-point-free rational automorphism Φ of period 3. Then G is nilpotent.*

Proof. By Lemma 4 for all $x \in G$, $\Phi^2(x)\Phi(x)x = e$ and hence also $\Phi^2(x^{-1})\Phi(x^{-1})x^{-1} = e$, i.e. $x\Phi(x)\Phi^2(x) = e$. Therefore $\Phi^2(x^{-1}) = x\Phi(x) = \Phi(x)x$, i.e. for all $x \in G$, x commutes with $\Phi(x)$. Let K be the subgroup of G generated by x and $\Phi(x)$; K is a commutative group stable under Φ and so by Lemma 2', $\bar{\Phi}$ is surjective on K. **Hence $\bar{\Phi}$ is surjective on G.**

Let B_0 be a Borel subgroup of G; then $\Phi(B_0)$ is also a Borel subgroup, hence by conjugacy $\Phi(B_0) = xBx^{-1}$. Since $\bar{\Phi}$ is surjective, $x = \bar{\Phi}(y)$ for some $y \in G$. Then it follows that $B = y^{-1}B_0y$ is a Borel subgroup of G stable under Φ. By Theorem 1', B is nilpotent and hence $G = B$ is nilpotent.

q. e. d.

THEOREM 2. *Let G be a semi-simple algebraic group and Φ a rational automorphism of G. Then Φ has a non-trivial fixed-point.*

Proof. We assume Φ acts fixed-point freely on G and arrive at a contradiction. The automorphisms of the semi-simple algebraic groups are known to be of the form $\Phi(x) = ax^\sigma a^{-1}$ where σ is an "outer" automorphism and is of finite order, $\sigma^r = 1$. When G is simple and $r > 1$ we may have either $r = 2$ (type A_n, D_n, E_6) or $r = 3$ (type D_4). By repeated application of Φ we have $\Phi^r(x) = bxb^{-1}$ where $b = \Phi^{r-1}(a) \cdots \Phi(a)a = aa^\sigma \cdots a^{\sigma^{r-1}}$, hence $b^\sigma = a^{-1}ba$. Then $\Phi(b) = ab^\sigma a^{-1} = b$ and so $b = e$ and $\Phi^r = 1$.

Let G' be one of the simple components of G; since the simple components of G are uniquely determined, $\Phi^i(G')$ is a simple component of G for each i, $0 \leqq i < r$. They cannot all be distinct for otherwise if $x \in G'$ the representation $\Phi^{r-1}(x) \cdots \Phi(x)x = e$ given by Lemma 4 would imply $x = e$. Therefore for some i, $0 < i < r$, $\Phi^i(G') = G'$. Let $d = (i, r)$ and $ai + br = d$; then $\Phi^d = \Phi^{ai}$, hence $\Phi^d(G') = G'$. Choose $d \mid r$ such that $\Phi^d(G') = G'$ and $G', \Phi(G'), \cdots, \Phi^{d-1}(G')$ are all distinct. Let $x \in G'$ be such that $\Phi^d(x) = x$; then if $y = \Phi^{d-1}(x) \cdots \Phi(x)x, \Phi(y) = y$ and so $y = e$ which implies $x = e$. Therefore Φ^d is a fixed-point-free automorphism of G' of order r/d. Replacing G by G' and Φ by Φ^d we have reduced our considerations to the case of a simple group G. But in this case $r = 2$ or 3 and then Proposition 2 and Lemma 7 imply G must be nilpotent, which is impossible since G is simple.

q. e. d.

THEOREM 3'. *Let G be a connected algebraic group admitting a fixed-point-free rational automorphism Φ. Then G is solvable. Moreover, if Φ is of finite order n prime to p, G is unipotent and if Φ is of order a power of p, G is a torus.*

Proof. Let R be the radical of G; R is a connected solvable characteristic algebraic subgroup of G. By Lemma 3′ the semi-simple algebraic group G/R admits a fixed-point-free automorphism so by Theorem 2, $G = R$ is solvable. The remaining assertion of the theorem now follows immediately from Theorem 1′. q. e. d.

5. A second structure theorem. Our first structure theorem (Theorem 1) asserts that if G is a Frobenius algebraic group such that G_0 is solvable then G_0 is nilpotent. The results of the preceding section allow us to remove the assumption of solvability.

THEOREM 3. *Let G be a Frobenius algebraic group. Then G_0 is nilpotent. Moreover, either G_0 is a torus of dimension $d \equiv 0 \pmod{p-1}$ and H is a p-group, or G_0 is unipotent and the order of H is prime to p.*

Proof. Let l be a prime dividing the order of H and let h be an element of H of order l. Since $hG_0h^{-1} = G_0$ and no element of G_0 centralizes h the map $\operatorname{ad} h\colon G_0 \to G_0$ defined by $\operatorname{ad} h(g) = hgh^{-1}$ is a fixed-point-free automorphism of G_0 of period l. By Theorem 3′, G_0 is nilpotent and the assertion of the theorem is an immediate consequence of Theorem 1. q. e. d.

6. The regular subgroup of a Frobenius algebraic group. Recall that if G is a Frobenius group we denote $M = \{G - \bigcup_{x \in G} xHx^{-1}\} \cup \{e\}$ and a theorem of Frobenius asserts M is a normal subgroup of G in the case G finite. We are now in a position to extend this theorem to algebraic groups.

THEOREM 4. *If G is a Frobenius algebraic group then M is a normal algebraic subgroup, called the regular subgroup of G.*

Proof. Let ρ be a rational representation of G with kernel G_0, $\bar{G} = \rho(G)$, $\bar{H} = \rho(H)$. Since G_0 is nilpotent (Theorem 3) Proposition 1 and Lemma 3 imply $\bar{G}$ is a Frobenius algebraic group. Since $\bar{G}$ is finite, $\bar{M}$ is a normal subgroup of $\bar{G}$. Let $g \in G$, $g \notin M$; then there exists $h \in H$, $h \neq e$, and $x \in G$ *such that* $g = xhx^{-1}$ and so $\bar{g} = \rho(g) \in \bigcup_{\bar{x} \in \bar{G}} \bar{x}\bar{H}\bar{x}^{-1}$. Moreover, since $G_0 \cap H = \{e\}$, $G_0 \cap \bigcup_{x \in G} xHx^{-1} = \{e\}$, hence $g \notin G_0$, i. e. $\bar{g} \neq \bar{e}$. Therefore $\bar{g} \notin \bar{M}$ and $M \supset \rho^{-1}(\bar{M})$.

Suppose now $g \in M$; if $g \in G_0$ then $\bar{g} = \rho(g) = \bar{e} \in \bar{M}$. If $g \notin G_0$ and if $\bar{g} \in \bigcup_{\bar{x} \in \bar{G}} \bar{x}\bar{H}\bar{x}^{-1}$ there exists $x \in G$ and $h \in H$, $h \neq e$, such that $hx^{-1}gx \in G_0$. By Lemmas 1a and 2, σ_h is surjective on G_0 and there exists $g_0 \in G_0$ such that $hx^{-1}gx = \sigma_h(g_0) = hg_0h^{-1}g_0^{-1}$, i. e. $g = xg_0h^{-1}g_0x^{-1} \in \bigcup_{x \in G} xHx^{-1}$ which is

impossible since $g \in M$. Therefore $\bar{g} \notin \bigcup_{\bar{x} \in \bar{G}} \bar{x}\bar{H}\bar{x}^{-1}$, i.e. $\bar{g} \in \bar{M}$ and $M \subset \rho^{-1}(\bar{M})$. Consequently $M = \rho^{-1}(\bar{M})$ and therefore M is a normal subgroup of G.

q.e.d.

THEOREM 5. *Let G be a Frobenius algebraic group and M the regular subgroup of G. Then $G = H \cdot M$, M_0 is nilpotent and either $M = M_s$ and H is a p-group or M_0 is unipotent and the order of H is prime to p. In any case, M_0 and M/M_0 are nilpotent and so M is solvable.*

Proof. By Theorem 3, G_0 is nilpotent and so by Lemma 3, if ρ is a rational representation of G with kernel G_0 then $\rho(G)$ is a finite Frobenius group (with regular subgroup $\rho(M)$). In the finite case then we have easily $\rho(G) = \rho(H) \cdot \rho(M)$. By the corollary to Proposition 1, $G_0 \subset M$ and it follows that $G = HM$; since $H \cap M = \{e\}$, $G = H \cdot M$. Also since $G_0 \subset M$, $M_0 = G_0$ and the next assertion follows from Theorem 3. By Thompson's theorem [4] M/M_0 is nilpotent. q.e.d.

7. An application. As a consequence of his theorem Thompson was able to prove a theorem concerned with maximal subgroups of a finite group [4]. The following is an extension to algebraic groups.

THEOREM 6. *Let G be an algebraic group and suppose that one of the maximal subgroups M of G is nilpotent. Then G_0 is solvable. If $[M : M_0]$ is odd, then G is solvable.*

Proof. If M is nilpotent so is $\bar{M}$ (smallest algebraic group containing M) and since M is maximal either $M = \bar{M}$ or $\bar{M} = G$; therefore we may assume M to be an algebraic subgroup of G. Since $M \subset N_G(M_0)$ (normalizer in G of M_0) either M_0 is normal in G or $M = N_G(M_0)$. Suppose first $M = N_G(M_0)$; then $M \supset N_{G_0}(M_0) = M \cap G_0 \supset M_0$, hence M_0 is a nilpotent subgroup of G_0 equal to its connected normalizer in G_0. Therefore M_0 is a Cartan subgroup of G_0 and is maximal nilpotent; since $M \cap G_0$ is nilpotent either $M \cap G_0 = G_0$, in which case G_0 is nilpotent, or else, as we now assume, $M_0 = M \cap G_0 = C_{G_0}(T)$ (centralizer in G_0 of T) where T is a maximal torus of G_0. Note $M \subset N_G(T)$ and if $N_G(T) = G$, $N_{G_0}(T) = N_G(T) \cap G_0 = G_0$ hence T is normal in G_0 and G_0 is nilpotent; we assume then $M = N_G(T)$, hence $M_0 = M \cap G_0 = N_{G_0}(T) = C_{G_0}(T)$ so the Weyl group of G_0 is reduced to $\{e\}$, hence G_0 is solvable.

Now suppose M_0 normal in G; by considering a rational representation of G with kernel M_0 we are reduced to the case M finite. We proceed by induction on the order m of M, the case $m = 1$ being trivial. In particular, the induction hypothesis allows us to assume if $N \neq \{e\}$ is a normal sub-

group of M then $N_G(N) = M$. Since $M \cap G_0 = N$ is normal in M, if $N \neq \{e\}$ we have $N_G(N) = M$ and $N_{G_0}(N) = M \cap G_0 = N$. Since N is nilpotent its center $Z(N) \neq \{e\}$; if $x \in Z(N)$, $x \neq e$, $C_G(x) \subset M$ hence $C_{G_0}(x) = N$. If $s \neq e$ is a semisimple element of $Z(N)$ then $s \in T$, T a maxmal torus of G_0 and $T \subset C_{G_0}(s) = N$ which is finite; therefore $T = \{e\}$ and G_0 is nilpotent. If $u \neq e$ is a unipotent element of $Z(N)$ then $u \in B_u$, B a Borel subgroup of G_0 and $Z(B_u) \subset C_{G_0}(u) = N$; therefore $B_u = \{e\}$ (since it is a connected nilpotent group and if $\dim B_u > 0$, $\dim Z(B_u) > 0$) and B, hence G_0, is a torus.

Finally then we may assume $M \cap G_0 = N = \{e\}$. Let $H = Z(M)$; since M is nilpotent $H \neq \{e\}$. Let $G' = G_0 H$ and suppose $x \in G'$ such that $xHx^{-1} \cap H \neq \{e\}$; say $x = gh$, $g \in G_0$, $h \in H$ and there exists $h' \in H$, $h' \neq e$, such that $gh'g^{-1} \in H$. Then $h'^{-1}gh'g^{-1} \in G_0 \cap H = \{e\}$, hence $h'^{-1}gh'g^{-1} = e$, i.e. $gh'g^{-1} = h'$ and $g \in C_G(h') \subset M$. But $g \in G_0$ and $G_0 \cap M = \{e\}$, so that $g = e$ and $x = h \in H$. Therefore G' is a Frobenius group and by Theorem 3, $G'_0 = G_0$ is nilpotent.

Now let ρ be a rational representation of G with kernel G_0, $\rho(G) = \bar{G}$, $\rho(M) = \bar{M}$. Then $\bar{G}$ is a finite group, $\bar{M}$ a maximal subgroup of $\bar{G}$ which is nilpotent, and the order of $\bar{M}$ divides $[M : M_0]$. Therefore if $[M : M_0]$ is odd so is the order of $\bar{M}$ and it follows from [4] that $\bar{G}$ is solvable. Then G, being an extension of a solvable group by a solvable group, is solvable.

q. e. d.

Actually the structure of $\bar{G} = G/G_0$ can be completely determined also in the case $[M : M_0]$ even because Thompson has determined the structure of all finite groups having a maximal subgroup which is nilpotent [5].

CORNELL UNIVERSITY.

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A FAMILY OF SIMPLE GROUPS ASSOCIATED WITH THE SIMPLE LIE ALGEBRA OF TYPE (G_2).*

By RIMHAK REE.[1]

Introduction. In this paper we construct a family of simple groups. The family contains finite as well as infinite groups. The order of the finite groups in the family are

$$q^3(q-1)(q^3+1), \qquad \text{where } q=3^{2n+1}, \qquad n=1,2,3,\cdots.$$

By using the fact that the above orders are divisible by 8 but not by 16, one can show easily that none of the simple groups listed in [1] or [4] has any of the above orders.[2]

The construction is carried out by applying to the Chevalley groups of type (G_2) a method which emerges naturally when one interprets Suzuki's [5] construction of his simple groups from a Lie theoretical point of view. The Lie theoretical interpretation of Suzuki groups is as follows. Let $\mathfrak{g}$ be the simple Lie algebra of type (B_2) over the complex number field, and $\Sigma=\{\pm a, \pm b, \pm(a+b), \pm(2a+b)\}$ the set of roots of $\mathfrak{g}$. For an arbitrary field K, define the algebra $\mathfrak{g}_K$ over K and the automorphisms $x_r(t)$, where $r\in\Sigma$, $t\in K$, of $\mathfrak{g}_K$ as in [2], and let G be the group generated by all the $x_r(t)$. Now assume that K is a field of characteristic 2 which admits an automorphism $t\to t^\theta$ such that $2\theta^2=1$. By using a representation of G (given, for example, in [3]) one can show easily that the matrix $S(t,u)$ given by Suzuki in [5] corresponds to $\alpha(t)\beta(u)$, where

$$\alpha(t)=x_a(t^\theta)x_b(t)x_{a+b}(t^{2\theta+1}); \qquad \beta(t)=x_{a+b}(t)x_{2a+b}(t^{2\theta}).$$

The above expressions lead one to look for an automorphism σ of G such that the elements $\alpha(t)\beta(u)$ are exactly those elements in $\mathfrak{U}$ (the subgroup of G generated by all $x_r(t)$ with $r>0$) which are left invariant by σ. It turns out that there exists indeed such an automorphism of G. It is given by

$$x_{\pm a}(t)\leftrightarrow x_{\pm b}(t^\theta), \qquad x_{\pm(a+2b)}(t)\leftrightarrow x_{\pm(a+b)}(t^\theta).$$

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[1] This work was done while the author held a Research Associateship, 1959-1960, of the Office of Naval Research, U. S. Navy.

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[2] See Theorems 8.3 and 8.5.

Denoting by $\mathfrak{B}$ the group generated by all $x_r(t)$ with $r < 0$, let $\mathfrak{U}^1$ (resp. $\mathfrak{B}^1$) be the group of elements in $\mathfrak{U}$ (resp. $\mathfrak{B}$) left invariant by σ. Then the group G^1 generated by $\mathfrak{U}^1$ and $\mathfrak{B}^1$ is simple when K has more thai 2 elements, and yields Suzuki groups when K is finite.

The Chevalley group G of type (G_2) happens to have a similar automorphism when K is a field of characteristic 3 which admits an automorphism $t \to t^\theta$ such that $3\theta^2 = 1$. Hence we can define the group G^1 in exactly the same way as above. As one of the main purposes of this paper we shall show that the group G^1 is simple when K has more that 3 elements.

In the case of Suzuki groups, G^1 turns out to be the group of *all* elements in G fixed by σ. For the groups of type (G_2) the same holds true when the field K is finite, but it is unknown whether or not one can, in general, dispense with the finiteness condition. Some efforts in this direction may be found in Section 8.

In the last section, we show that any automorphism of the *finite* simple group G^1 of type (G_2) can be written as the product of an inner automorphism and an automorphism induced by an automorphism of the ground field. The author has been unable to extend this result to the case of infinite groups.[3]

1. Chevalley groups. Although we shall use later only Chevalley groups of type (G_2), we treat here the Chevalley groups in general, since it does not make the exposition more complicated.

Let Σ be a simple root system. We fix an order in Σ so that we can talk about positive, negative, and fundamental roots. Let P (denoted in [2] by P_r) be the additive group generated by Σ. For r, s in Σ define the integer $s(r)$ by $s(r) = p - q$, where q (resp. $-p$) is the maximum (resp. minimum) integer i such that $s + ir$ is a root. Then for each $r \in \Sigma$, the map $s \to s(r)$ can be extended to a linear map on P. The symmetry w_r with respect to $r \in \Sigma$ is the permutation of Σ defined by $w_r(s) = s - s(r)r$. The Weyl group W of Σ is the group generated by all the w_r, $r \in \Sigma$.

Now we fix an arbitrary field K, and denote by K^* the multiplicative group of K. For $r \in \Sigma$ and $z \in K^*$, define the homomorphism $\chi_{r,z}: P \to K^*$ by $\chi_{r,z}(u) = z^{u(r)}$. Denote by X the group generated by all the $\chi_{r,z}$, where $r \in \Sigma$, $z \in K^*$. It can be shown that $w(s)(w(r)) = s(r)$ for any $r, s \in \Sigma$ and $w \in W$. From this it follows that $\chi \circ w \in X$ for any $\chi \in X$ and $w \in W$. It is shown in [2] that, for any pair of Σ and K, there exists a group G (denoted

[3] J. Tits remarks that the finiteness condition seems to be dispensed with if one uses his geometric interpretation of the group G^1 given in [7].

by G' in [2]), which we shall call the *Chevalley group* of type Σ over K, and elements $x_r(t)$, where $r \in \Sigma$, $t \in K$, in G such that the conditions (1.1)-(1.8), below, are satisfied:

(1.1) G is generated by the elements $x_r(t)$, where $r \in \Sigma$, $t \in K$.

(1.2) For any two positive roots r and s in Σ,

we have

$$x_r(t)x_s(u)x_r(-t) = x_s(u) \prod_{i,j} x_{ir+js}(C_{ij;r,s}t^i u^j),$$

where the product is taken over all pairs (i, j) of integers such that $ir + js$ is a root, the pairs being arranged such that the roots $ir + js$ form an increasing sequence, and where $C_{ij;r,s}$ are certain integral constants depending only on Σ (not on K).

(1.3) For each $r \in \Sigma$, there exists a homomorphism $\phi: SL(2; K_Q \to G$ such that

$$\phi_r \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = x_{-r}(t), \qquad \phi_r \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = x_r(t), \qquad (t \in K).$$

(1.4) There exists a homomorphism $h: X \to G$ such that

$$h(\chi_{r,z}) = \phi_r \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$$

for all $r \in \Sigma$ and $z \in K^*$.

(1.5) For any $\chi \in X$ and $r \in \Sigma$, we have

$$h(\chi)x_r(t)h(\chi)^{-1} = x_r(\chi(r)t), \qquad (t \in K).$$

(1.6) For

$$\omega_r = \phi_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

we have

$$\omega_r x_s(t)\omega_r^{-1} = x_{w_r(s)}(\eta_{r,s}t), \qquad (s \in \Sigma, t \in K),$$

where $\eta_{r,s}$ are certain integral constants ± 1 depending only on Σ (not on K).

(1.7) For any $r_1, \cdots, r_m \in \Sigma$ and any $\chi \in X$, we have

$$\omega h(\chi)\omega^{-1} = h(\chi')$$

where

$$\omega = \omega_{r_1} \cdots \omega_{r_m}$$

and where $\chi' \in X$ is defined by $\chi'(s) = \chi(w^{-1}(s))$ with

$$w = w_{r_1} \cdots w_{r_m}.$$

(1.8) If $r_1, \cdots, r_m \in \Sigma$ and $\chi \in X$ satisfy $\eta_s = \chi(s)$ for all $s \in \Sigma$, and if

$$w_{r_1} \cdots w_{r_m} = 1,$$

then

$$\omega_{r_1} \cdots \omega_{r_m} = h(\chi).$$

Here η_s are integers ± 1 defined from $r_1, \cdots, r_m$ by

$$\eta_s = \eta_{r_1, v_1(s)} \cdots \eta_{r_m, v_m(s)},$$

where

$$v_m = 1,\ v_{m-1} = w_{r_m},\ v_{m-2} = w_{r_{m-1}} w_{r_m}, \cdots,$$
$$v_1 = w_{r_2} w_{r_3} \cdots w_{r_m}.$$

Note that the above properties (1.2)-(1.8) are all expressed in the form of "relations" of the generators $x_r(t)$. Therefore, by using the constants $C_{ij;r,s}$ and $\eta_{r,s}$ which arise from a Chevalley group, we can consider a group G abstractly defined by the generators and the relations (1.2)-(1.8). In order to distinguish this group G from the underlying Chevalley group which gives rise to the constants $C_{ij;r,s}$ and $\eta_{r,s}$, we shall denote the latter by $\hat{G}$, and let $x \to \hat{x}$ be the natural homomorphism of G onto $\hat{G}$. We shall show, in the rest of this section, that $x \to \hat{x}$ is an isomorphism. First we shall derive some consequences of (1.1)-(1.8).

(1.9) If $r = -a$, where a is a fundamental root, and if s is a positive root, then we have an identity of the form given in (1.2).

Proof. Since a is fundamental, w_a maps every positive root $\neq a$ upon a positive root. Applying (1.2) to a and $w_a(s)$, transform the resulting identity by ω_a, and rearrange the factors on the right such that the roots form an increasing sequence. Then by (1.6) we have the resired result.

(1.10) For any $r \in \Sigma$, we have $\omega_{-r}\omega_r = 1$.

Proof. Since

$$\omega_{-r} = \phi_{-r}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \phi_{-r}\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)$$
$$= x_{-r}(1)x_r(-1)x_{-r}(1) = \phi_r\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right).$$

we have

$$\omega_{-r}\omega_r = \phi_r\left(\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)$$
$$= \phi_r\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = 1.$$

(1.11) If $r_1, \cdots, r_m \in \Sigma$ satisfy

$$(*) \qquad w_{r_1}\cdots w_{r_m} = 1,$$

then there exists $\chi \in X$ such that $\chi(s) = \eta_s$ for all $s \in \Sigma$, where η_s are the integers defined in (1.8) from $r_1, \cdots, r_m$.

Proof. Let

$$\omega = \omega_{r_1}\cdots\omega_{r_m}.$$

Then by (1.6) and the definition of η_s we have

$$\omega x_s(t)\omega^{-1} = x_s(\eta_s t)$$

for all $t \in K$. Applying the homomorphism $x \to \hat{x}$ to the above,

$$\hat{\omega}\hat{x}_s(t)\hat{\omega}^{-1} = \hat{x}_s(\eta_s t).$$

On the other hand, it is known ([2], Lemma 3, p. 37; Lemma 4, p. 49) that (*) implies that $\hat{\omega} = \hat{h}(\chi)$ for some $\chi \in X$. Then

$$\hat{h}(\chi)\hat{x}_s(t)\hat{h}(\chi)^{-1} = \hat{x}_s(\chi(s)t) = \hat{x}_s(\eta_s t).$$

Hence we have $\chi(s) = \eta_s$ for all $s \in \Sigma$.

From (1.8) and (1.11) we obtain

(1.12) If $r_1, \cdots, r_m \in \Sigma$ satisfy the condition (*) in (1.11), then for some $\chi \in X$ we have

$$\omega_{r_1}\cdots\omega_{r_m} = h(\chi).$$

Denote by $\mathfrak{U}$ (resp. $\mathfrak{V}$) the subgroup of G generated by the $x_r(t)$ with $r > 0$ (resp. $r < 0$), and by $\mathfrak{H}$ the group generated by all the $h(\chi)$, $\chi \in X$. For each $w \in W$ denote by $\mathfrak{U}_w$ the group generated by all the $x_r(t)$, $t \in K$, such that $r > 0$ and $w(r) < 0$. Also for each $w \in W$, choose once and for all roots $r_1, \cdots, r_m \in \Sigma$ such that

$$w = w_{r_1}\cdots w_{r_m}$$

and set

$$\omega(w) = \omega_{r_1}\cdots\omega_{r_m}.$$

Then we have the following fundamental proposition:

(1.13) Every element in G is written uniquely in the form $uh\omega(w)u'$, where $u \in \mathfrak{U}$, $h \in \mathfrak{H}$, $w \in W$, and $u' \in \mathfrak{U}_w$.

First we shall prove the uniqueness part of (1.13). Suppose $uh\omega(w)u' = u_1h_1\omega(w_1)u_1'$ with $u_1 \in \mathfrak{U}$, $h_1 \in \mathfrak{H}$, $w_1 \in W$, and $u_1' \in \mathfrak{U}_{w_1}$. Then $\hat{u}\hat{h}\hat{\omega}(w)\hat{u}' = \hat{u}_1\hat{h}_1\hat{\omega}(w_1)\hat{u}_1'$. Then by [2], Theorem 2, we have $\hat{u} = \hat{u}_1$, $\hat{h} = \hat{h}_1$, $\hat{w} = \hat{w}_1$, $\hat{u}' = \hat{u}_1'$. By (1.2) we can write u and u_1 in the form $\prod_r x_r(t_r)$, where r runs over all positive roots arranged as an increasing sequence and where $t_r \in K$. Then by [2], Lemma 6, p. 39, $\hat{u} = \hat{u}_1$ implies that t_r for u and u_1 coincide for all $r > 0$. Hence $u = u_1$ Similarly, $u' = u_1'$. Since $w = w_1$, we have $h = h'$ also. Thus the uniqueness part of (1.13) is proved.

Before continuing the proof of (1.13) we shall make a definition. For any element ω of the form

$$\omega = \omega_{r_1} \cdots \omega_{r_m} \tag{1.14}$$

define $\zeta(\omega)$ by

$$\zeta(\omega) = w_{r_1} \cdots w_{r_m}.$$

It can be easily seen from (1.6) and the uniqueness part of (1.13), above, that $\zeta(\omega)$ is independent of the mode of representing ω.

Now, a process of putting the given element x in G in the form of (1.13) is given in [2], p. 38-40. The properties (1.1)-(1.9) are sufficient to carry out the process. Since this process is needed later, we shall describe it in detail for the convenience of the reader. It consists of the three steps (1.15)-(1.17), below.

(1.15) *Step* 1. We express the given element x as $x_1x_2 \cdots x_n$, where each factor x_i is either of the form $x_{\pm a}(t)$ or of the form $\omega_{\pm a}$ with a fundamental root a.

This step may be divided into the following 2 parts:

(i) Write the given element x in the form $\prod_r x_r(t_r)$.

(ii) For each non-fundamental $r \in \Sigma$ appearing in (i), above, choose $w \in W$ and a fundamental root a such that $r = w(a)$; find fundamental roots $a_1, \cdots, a_m$ such that

$$w = w_{a_1} \cdots w_{a_m},$$

and replace $x_r(t)$ in (i) by

$$\omega_{a_1} \cdots \omega_{a_m} x_a(\pm t) \omega_{-a_m} \cdots \omega_{-a_1}.$$

(1.16) *Step* 2. We express the given element x in the form

$$(*) \qquad x = uh\omega u'$$

where $u, u' \in \mathfrak{U}$, $h \in \mathfrak{H}$ and ω is an element of the form given in (1.14).

This step is carried out as follows. Let $x = x_1 \cdots x_n$ as in (1.15), and set

$$y_0 = 1, y_1 = x_n, y_2 = x_{n-1}x_n, \cdots, y_n = x_1 \cdots x_n.$$

Then y_0 is trivially in the form (*). Suppose $y_i = uh\omega u'$ is in the form (*). If $y_{i+1} = x_a(t)y_i$ with a fundamental root a, then y_{i+1} is also in the form (*). If $y_{i+1} = \omega_a y_i$ or if $y_{i+1} = x_{-a}(t)y_i$, then write u appearing in y_i in the form $u = u_1 x_a(t_1)$, where u_1 is a product of some $x_r(t')$ with $r > 0$, $r \neq a$. Then by [2], Lemma 8, p. 40, y_{i+1} can be written in the form $u_2 hv\omega u'$, where $v \in \phi_a(SL(2;K))$. Set $w = \zeta(\omega)$. Two possibilities arise: $w^{-1}(a) < 0$.

Case 1. $w^{-1}(a) > 0$. By [2], p. 34, write v as $v = x_a(t')h$ or $v = x_a(t')h\omega_a x_a(t'')$, where $h \in \mathfrak{H}$. Then it can be seen easily that y_{i+1} can be put in the form (*).

Case 2. $w^{-1}(a) < 0$. Use $\omega_1 = \omega_a\omega$ instead of ω. Set $w_1 = \zeta(\omega_1)$. Then $w_1(a) < 0$. Hence we are in case 1.

Thus, proceeding inductively, we can put $y_n = x$ in the form (*).

(1.17) *Step* 3. We finally express the given element x in the form specified in (1.13), starting from the expression $x = uh\omega u'$ obtained in (1.16). By making repeated use of (1.2), if necessary, we express $u' \in \mathfrak{U}$ as $u' = u_1 u_2$, where u_1 is a product of $x_r(t)$ such that $r > 0$, $w(r) > 0$, where $w = \zeta(\omega)$, while $u_2 \in \mathfrak{U}_w$. Then by (1.5), (1.6), and (1.7), we can write $x = u_3 h\omega u_2$ with $u_3 \in \mathfrak{U}$. Also, by (1.12), $\omega = h_1\omega(w)$ with $h_1 \in \mathfrak{H}$. Thus (1.13) is proved.

Now let x be any element in the kernel of the homomorphism $x \to \hat{x}$, and write $x = uh\omega(w)u'$ as in (1.13). Then arguing as in the proof of the uniqueness part of (1.13), one can show easily that $\hat{x} = 1$ implies $x = 1$. Hence the map $x \to \hat{x}$ is an isomorphism of G onto $\hat{G}$. Thus we have proved

THEOREM 1.18. *For any simple root system* Σ *and any field* K, *the Chevalley group* G *of type* Σ *over* K *is characterized by the properties* (1.1)-(1.8).[4]

The constants $C_{ij;r,s}$ and $\eta_{r,s}$ are not uniquely determined by Σ. If, how-

[4] For another characterization of Chevalley groups by generators and defining relations, see a forthcoming book by J. Tits.

ever, we have, in the simple Lie algebra of type Σ over the complex number field, a set $\{X_r \mid r \in \Sigma\}$ of root vectors which satisfies the conditions in Theorem 1 of [2], then we can obtain a set of the above constants by setting $x_r(t) = \exp(t \operatorname{ad} X_r)$ and by the relations (1.2) and (1.6). The constants $\eta_{r,s}$ can actually be defined by ([2], p. 36)

$$\omega_r X_s = \eta_{r,s} X_{w_r(s)}. \tag{1.19}$$

We have

$$\eta_{r,r} = -1, \quad \eta_{r,-s} = \eta_{r,s}, \quad \eta_{-r,s} = \eta_{r,w_r(s)}, \tag{1.20}$$

the first of which follows from (1.19) and the relation $\omega_r = x_r(1)x_{-r}(-1)x_r(1)$, the second from [2], p. 36, and the third from (1.9). Also from (1.9) we have

(1.21) If $r, s \in \Sigma$ are distinct, and if none of $r+s$, $r-s$ are in Σ, then $\eta_{r,s} = 1$.

For $r, s \in \Sigma$ such that $r+s \in \Sigma$, define the number $N_{r,s}$ by $[X_r, X_s] = N_{r,s} X_{r+s}$. Then from (1.19) we have

(1.22) If $r, s \in \Sigma$ and if $\pm r$, $\pm s$, $\pm (r+s)$ are the only linear combinations of r, s which are in Σ, then $\eta_{r,s} = N_{r,s}$.

(1.23) For any $r, s, u \in \Sigma$ such that $s+u \in \Sigma$, we have

$$\eta_{r,s}\eta_{r,u} N_{w_r(s),w_r(u)} = \eta_{r,s+u} N_{s,u}.$$

2. The root system of type (G_2). The root system Σ of type (G_2) can be given as

$$\Sigma = \{\pm \xi_i, \xi_i - \xi_j \mid i, j = 1, 2, 3; i \neq j\},$$

where $\xi_1 + \xi_2 + \xi_3 = 0$. The Weyl group W of Σ is of order 12, and consists of elements w such that $w(\xi_i) = e\xi_{\pi(i)}$, $i = 1, 2, 3$, and $e = \pm 1$, and where π is a permutation of $1, 2, 3$. We shall write $w = (e, \pi)$. Then $w_r = (1, (ij))$ for $r = \xi_i - \xi_j$, where (ij) denotes the transposition of i and j, and $w_r = (-1, (i'i''))$ for $r = \pm \xi_i$, where i, i', i'' is a permutation of 1, 2, 3.

The relationship between the roots and the action of W on Σ can be seen easily in Fig. 1, where Σ is represented as a system of vectors on a Euclidean plane. The element $w_r \in W$ is nothing but the symmetry with respect to the line which passes through the origin and which is perpendicular to the vector representing the root $r \in \Sigma$. The function $\lambda(r)$ on Σ defined by

$$\lambda(\pm \xi_i) = 1, \qquad \lambda(\xi_i - \xi_j) = 3$$

for all $i, j = 1, 2, 3$; $i \neq j$, has a clear meaning in Fig. 1.

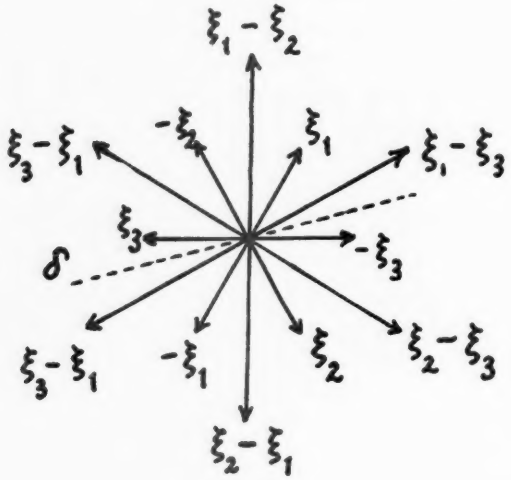

FIGURE 1.

Since $w_r(s) = s - s(r)r$, one can see easily that

$$(r, s) = \tfrac{1}{2}\lambda(s)r(s) \tag{2.1}$$

coincides with the Euclidean inner product of the vectors $r, s \in \Sigma$.

Denote by P, as in Section 1, the additive group generated by Σ, and define a homomorphism $\psi: P \to P$ by

$$\psi(\xi_1) = \xi_2 - \xi_3, \qquad \psi(\xi_2) = \xi_1 - \xi_2. \tag{2.2}$$

Then one can see easily that

$$\psi(r) = \lambda(r)\bar{r} \tag{2.3}$$

holds for all $r \in \Sigma$, where $r \to \bar{r}$ is a permutation of order 2 of Σ such that

$$\overline{-r} = -\bar{r}, \qquad \lambda(r)\lambda(\bar{r}) = 3 \tag{2.4}$$

for all $r \in \Sigma$. From the definition of ψ, it is clear that $\psi = \sqrt{3}\psi_\delta$, where ψ_δ is the symmetry with respect to the line δ bisecting the angle between $\xi_1 - \xi_3$ and $-\xi_3$ in Fig. 1.[5] Hence $(\psi(r), \psi(s)) = 3(r, s)$, which yields, by (2.1), (2.3), and (2.4),

$$\lambda(r)\bar{r}(\bar{s}) = \lambda(s)r(s) \tag{2.5}$$

for all r, s in Σ.

[5] The idea of considering the line δ in connection with the map ψ was suggested to the author by J. Tits.

Applying ψ to $w_r(s) = s - s(r)r$, we get

$$\psi(w_r(s)) = \lambda(s)\bar{s} - \lambda(r)s(r)\bar{r} = \lambda(s)\bar{s} - \lambda(s)\bar{s}(\bar{r})\bar{r}$$

by (2.5). Since $\lambda(w_r(s)) = \lambda(s)$, we have from the above

$$\overline{w_r(s)} = w_{\bar{r}}(\bar{s}) \tag{2.6}$$

for all $r, s \in \Sigma$. For each $w \in W$, define a permutation $\bar{w}$ of Σ by

$$\bar{w}(\bar{s}) = \overline{w(s)}. \tag{2.7}$$

Then from (2.6) we have immediately

(2.8) The map $w \to \bar{w}$ is an automorphism of the Weyl group W such that $\bar{w}_r = w_{\bar{r}}$ for all $r \in \Sigma$.

We shall consider elements invariant under the automorphism $w \to \bar{w}$. Suppose $w = (e, \pi)$ satisfies $\bar{w} = w$. Then from (2.7) we obtain

$$\bar{\xi}_{\pi(1)} = \xi_{\pi(2)} - \xi_{\pi(3)}, \quad \bar{\xi}_{\pi(2)} = \xi_{\pi(1)} - \xi_{\pi(2)}$$

from which one can deduce easily that π is the identity permutation. Hence

(2.9) If $\bar{w} = w$, then $w = 1$ or $w = w_0$, where w_0 is defined by $w_0(r) = -r$ for all $r \in \Sigma$.

3. Chevalley groups of type (G_2). We shall determine a set of the constants $C_{ij;r,s}$ and $\eta_{r,s}$ for the root system Σ of type (G_2). The simple Lie algebra $\mathfrak{g}$ of type (G_2) has a basis

$$X_{ij}, \quad X_{i0}, \quad X_{0i}, \quad H_{ij}, \tag{3.1}$$

where $i, j = 1, 2, 3$; $i \neq j$, and where

$$H_{ij} + H_{ji} = 0; \quad H_{12} + H_{23} + H_{31} = 0. \tag{3.2}$$

The multiplication table is

$$\begin{aligned}
&[X_{ij}X_{jk}] = X_{ik}, && [X_{i0}X_{0j}] = 3X_{ij},\\
&[X_{0i}X_{ij}] = X_{0j}, && [X_{ij}X_{j0}] = X_{i0},\\
&[X_{i0}X_{j0}] = 2(ijk)X_{0k}, && [X_{0i}X_{0j}] = -2(ijk)X_{k0},\\
&[X_{ij}X_{ji}] = H_{ij}, && [X_{i0}X_{0i}] = H_{ij} + H_{ik},\\
&[X_{i0}H_{ij}] = -X_{i0}, && [X_{0i}H_{ij}] = X_{0i},\\
&[X_{ij}H_{ij}] = -2X_{ij}, && [X_{ij}H_{jk}] = [X_{ij}H_{ki}] = X_{ij},
\end{aligned} \tag{3.3}$$

where i, j, k run over distinct values of 1, 2, 3, and where $(ijk) = 1$ or -1 according as i, j, k is an even or odd permutation of 1, 2, 3. All the other products of basis element which cannot be computed trivially from (3.2) and (3.3) are zero.

The elements H_{ij} span a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and X_{ij}, X_{i0}, X_{0i} are root vectors with respect to $\mathfrak{h}$, corresponding respectively to the roots $\xi_i - \xi_j$, ξ_i, $-\xi_i$ given in Section 2. These root vectors satisfy the conditions in Theorem 1 of [2], and hence the table (3.3) can be used to compute a set of the constants $C_{ij;r,s}$ and $\eta_{r,s}$.

The constants $\eta_{r,s}$ can be computed easily by (1.19)-(1.23), and by the fact that any even permutation of i, j, k leaves the table (3.3) invariant. The result of the computation is give in table (3.4).

(3.4) The table of $\eta_{r,s}$

r \ s	ξ_1	ξ_2	ξ_3	$\xi_1-\xi_2$	$\xi_2-\xi_3$	$\xi_3-\xi_1$
ξ_1	−1	−1	1	−1	1	1
ξ_2	1	−1	−1	1	−1	1
ξ_3	−1	1	−1	1	1	−1
$\xi_1-\xi_2$	−1	1	1	−1	1	−1
$\xi_2-\xi_3$	1	−1	1	−1	−1	1
$\xi_3-\xi_1$	1	1	−1	1	−1	−1

All the other value of $\eta_{r,s}$ which are not found in (3.4) can readily be supplied by (1.20).

As an immediate application of (3.4) and (1.20), we can verify the equality

$$\eta_{\bar{r},\bar{s}} = \eta_{r,s} \tag{3.5}$$

for all $r, s \in \Sigma$, where $r \to \bar{r}$ is the map defined in Section 2. Also, using (3.4) and (1.20) we can show that

$$\omega_0 = \omega_{\xi_1}\omega_{\xi_2-\xi_3} = \omega_{\xi_2-\xi_3}\omega_{\xi_1}$$

satisfies

$$\omega_0 x_r(t)\omega_0^{-1} = x_{-r}(-t) \tag{3.6}$$

for all $r \in \Sigma$ and $t \in K$, and hence

$$\omega_0^2 = 1. \tag{3.7}$$

Similarly, using (3.4) and the symmetry of the table (3.3), and noting that there can be only one element $\omega_0 \in G$ satisfying (3.6), one can show that

$$\begin{aligned} \omega_0 &= \omega_{\xi_i}\omega_{\xi_j-\xi_k} = \omega_{\xi_j-\xi_k}\omega_{\xi_i} \\ &= \omega_{-\xi_k}\omega_{\xi_j-\xi_i} = \omega_{\xi_j-\xi_i}\omega_{-\xi_k} \end{aligned} \tag{3.8}$$

holds for any even permutation i, j, k of 1, 2, 3.

Now, in order to consider the constants $C_{ij;r,s}$, we shall fix an order in Σ. Set $a=\xi_2$, $b=\xi_1-\xi_2$. Then all the roots are $\pm a$, $\pm b$, $\pm(a+b)$, $\pm(2a+b)$, $\pm(3a+b)$, $\pm(3a+2b)$. We can regard $a, b, a+b, \cdots$ to be positive. Then the fundamental roots are a and b. For the map $r\to\bar{r}$ defined in Section 2, we have

$$\bar{a}=b, \quad (a+b)^{-}=3a+b, \quad (2a+b)^{-}=3a+2b. \tag{3.9}$$

Now, the constants $C_{1j;r,s}$ and $C_{i1;r,s}$ can be computed easily from (3.3) using the formulas given in [2], p. 36, but $C_{3,2;a,b}$ must be computed directly from (3.3). We shall express the results of our computation in the following form:

$$\begin{aligned}
(x_a(t), x_b(u)) &= x_{a+b}(-tu)x_{2a+b}(-t^2u)x_{3a+b}(t^3u)x_{3a+2b}(-2t^3u^2),\\
(x_a(t), x_{a+b}(u)) &= x_{2a+b}(-2tu)x_{3a+b}(3t^2u)x_{3a+2b}(3tu^2),\\
(x_a(t), x_{2a+b}(u)) &= x_{3a+b}(3tu),\\
(x_b(t), x_{3a+b}(u)) &= x_{3a+2b}(tu),\\
(x_{a+b}(t), x_{2a+b}(u)) &= x_{3a+2b}(3tu),
\end{aligned} \tag{3.10}$$

where $(x,y)=x^{-1}y^{-1}xy$. For all the other pairs of positive roots r, s, we have $(x_r(t), x_s(u))=1$.

4. An automorphism of G. In this section we assume that K is a perfect field of characteristic 3, and that $t\to t^\theta$ is an automorphism of K.

Let Σ be the root system of type (G_2). Using the notation introduced in Section 2, define $\bar{\chi}$ for each $\chi\in X$, by

$$\bar{\chi}(r)=\chi(\psi(r))^\theta=\chi(\bar{r})^{\lambda(r)\theta}. \tag{4.1}$$

Then one can show easily by (4.1) and (2.5) that

$$\bar{\chi}_{r,s}=\chi_{\bar{r},z^{\lambda(\bar{r})\theta}} \tag{4.2}$$

for all $r\in\Sigma$ and $z\in K^*$.

Let G be the Chevalley group of type (G_2) over K. Set

$$\bar{x}_r(t)=x_{\bar{r}}(t^{\lambda(\bar{r})\theta}). \tag{4.3}$$

Define the homomorphism $\bar{\phi}_r: SL(2;K)\to G$ by

$$\bar{\phi}_r\begin{pmatrix} t_1 & t_2\\ t_3 & t_4\end{pmatrix}=\phi_{\bar{r}}\begin{pmatrix} t_1^{\lambda(\bar{r})\theta} & t_3^{\lambda(\bar{r})\theta}\\ t_2^{\lambda(\bar{r})\theta} & t_4^{\lambda(\bar{r})\theta}\end{pmatrix}. \tag{4.4}$$

Define the homomorphism $\bar{h}: X\to G$ by

$$\bar{h}(\chi)=h(\bar{\chi}). \tag{4.5}$$

We shall show that we can replace $x_r(t)$, ϕ_r, h in (1.1)-(1.8) by $\bar{x}_r(t)$, $\bar{\phi}_r$, $\bar{h}$, respectively, without losing the validity of the statements. Denote by (1.n′) the statement obtained from (1.n) by the replacement. Then (1.1′) is clear. (1.3′) follows easily from (4.3), (4.4) and (2.4). (1.4′) follows from (1.4), (4.2), and (4.4). (1.5′) follows from (1.5), (4.1), and (4.3). As for (1.6′), ω_r is replaced by $\omega_{\bar{r}}$, since

$$\bar{\phi}_r\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \phi_{\bar{r}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \omega_{\bar{r}}. \tag{4.6}$$

Then (1.6′) follows from (1.6), (4.3), (3.5), (2.6), and the fact that $\lambda(w(r)) = \lambda(r)$ for all $w \in W$ and $r \in \Sigma$. As for (1.7′), ω is, by (4.6), replaced by

$$\bar{\omega} = \omega_{\bar{r}_1} \cdots \omega_{\bar{r}_m}.$$

Hence, by (1.7) and (2.8), we have $\bar{\omega}\bar{h}(\chi)\bar{\omega}^{-1} = h(\chi_1)$, where χ_1 is defined by $\chi_1(s) = \bar{\chi}(\bar{w}^{-1}(s))$. Then, by (4.1),

$$\begin{aligned} \chi_1(s) &= \chi((\bar{w}^{-1}(s))^{-})^{\lambda(\bar{w}^{-1}(s))\theta} \\ &= \chi(w^{-1}(\bar{s}))^{\lambda(s)\theta} = \overline{\chi'}(s), \end{aligned}$$

where r^{-} denotes $\bar{r}$ for $r \in \Sigma$. Therefore, $\chi_1 = \overline{\chi'}$. Now, (1.7′) follows from (4.5). Also it can be seen that (1.8′) follows from (1.8), (2.8), (3.5), and (4.1). It remains to prove (1.2′). Since K is of characteristic 3, (3.10) now becomes

$$\begin{aligned} (x_a(t), x_b(u)) &= x_{a+b}(-tu)x_{2a+b}(-t^2u)x_{3a+b}(t^3u)x_{3a+2b}(t^3u^2), \\ (x_a(t), x_{a+b}(u)) &= x_{2a+b}(tu), \\ (x_b(t), x_{3a+2b}(u)) &= x_{3a+2b}(tu), \end{aligned} \tag{4.7}$$

and all the other commutators of the form $(x_r(t), x_s(u))$, where $r > 0$, $s > 0$, are 1. Since the set of pairs $\{a, b\}$, $\{a, a+b\}$, $\{b, 3a+b\}$ is invariant under the map $\{r, s\} \to \{\bar{r}, \bar{s}\}$, it is clear, in view of (4.3), that $(x_r(t), x_s(u)) = 1$, $r > 0$, $s > 0$, implies $(\bar{x}_r(t), \bar{x}_s(u)) = 1$. Since

$$\begin{aligned} \lambda(a) &= \lambda(a+b) = \lambda(2a+b) = 1, \\ \lambda(b) &= \lambda(3a+b) = \lambda(3a+2b) = 3, \end{aligned}$$

we have by (3.9)

$$\begin{aligned} (\bar{x}_a(t), \bar{x}_b(u)) &= (x_b(t^{3\theta}), x_a(u^\theta)) = (x_a(u^\theta), x_b(t^{3\theta}))^{-1} \\ &= x_{3a+2b}(-u^{3\theta}t^{6\theta})x_{3a+b}(-u^{3\theta}t^{3\theta})x_{2a+b}(u^{2\theta}t^{3\theta})x_{a+b}(u^\theta t^{3\theta}) \\ &= x_{3a+b}((-tu)^{3\theta})x_{3a+2b}((-t^2u)^{3\theta})x_{a+b}((t^3u)^\theta)x_{2a+b}((t^3u^2)^\theta) \\ &= \bar{x}_{a+b}(-tu)\bar{x}_{2a+b}(-t^2u)\bar{x}_{3a+b}(t^3u)\bar{x}_{3a+2b}(t^3u^2). \end{aligned}$$

Thus it is shown that the first equation of (4.7) remains valid under the replacement $x_r(t) \to \bar{x}_r(t)$. Similarly the rest of (1.2′) can be proved. Thus we have proved (1.1′)-(1.8′). Then from Theorem 1.18 and the simplicity of G, we obtain

THEOREM 4.8. *If K is a perfect field of characteristic 3, and if $t \to t^\theta$ is an automorphism of K, then the Chevalley group of type (G_2) over K admits an automorphism $x \to x^\sigma$ such that*

$$x_r(t)^\sigma = x_{\bar{r}}(t^{\lambda(\bar{r})\theta})$$

for all $r \in \Sigma$ and $t \in K$. The automorphism σ maps $\mathfrak{U}$, $\mathfrak{H}$, $\mathfrak{V}$ onto themselves, respectively, and

$$h(\chi)^\sigma = h(\bar{\chi}), \qquad \omega_r{}^\sigma = \omega_{\bar{r}}$$

for all $\chi \in X$ and $r \in \Sigma$, where $\bar{\chi}$ is defined by (4.1).[6]

5. Invariant elements. From now on we assume that K is a perfect field of characteristic 3, and that the automorphism θ satisfies

$$3\theta^2 = 1, \tag{5.1}$$

the right hand side of (5.1) being the identity automorphism.[7]

Before considering the group G, we shall derive a few properties of the field K from (5.1). Since

$$(z^{3\theta+1})^{3\theta-1} = z^2, \qquad (z^{3\theta-2})^{3\theta+2} = z^{-1},$$

we have

(5.2) Every element in K^{*2} is of the form $z^{3\theta-1}$ with $z \in K^*$.

(5.3) Every element in K^* is of the form $z^{3\theta-2}$ and also of the form $z^{3\theta+2}$.

(5.4)[8] -1 has no square root in K.

Proof. Suppose $z^2 = -1$. Then $(z^\theta)^2 = -1$, $z^\theta = \pm z$, and hence

$$z = (z^{3\theta})^\theta = (-z^\theta)^\theta = (\mp z)^\theta = -z,$$

but this is impossible.

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[6] After the completion of earlier versions of this paper and [9], the author learned from Tits that the existence of the automorphism σ was already mentioned in [6], p. 282. See also R. Steinberg [8].

[7] It was discovered by Tits that K need not be assumed to be perfect if one considers an endomorphism η, instead of θ, of K such that $\eta^3 = 3$. For the details of this, see [7].

[8] This lemma was pointed out to the author by R. Steinberg and J. Tits, independently. The proof given here is due to Tits.

An element x in G will be said to be σ-*invariant* if $x^\sigma = x$. First we shall consider σ-invariant elements in $\mathfrak{U}$. Suppose

$$x = x_a(t_{10})x_b(t_{01})x_{a+b}(t_{11})x_{2a+b}(t_{21})x_{3a+b}(t_{31})x_{3a+2b}(t_{32})$$

is σ-invariant. We have

$$x^\sigma = x_b(t_{10}{}^{3\theta})x_a(t_{01}{}^\theta)x_{3a+b}(t_{11}{}^{3\theta})x_{3a+2b}(t_{21}{}^{3\theta})x_{a+b}(t_{31}{}^\theta)x_{2a+b}(t_{32}{}^\theta).$$

By (4.7), the right hand side of the above can be changed into the form $x_a(s_{10})x_b(s_{01})\cdots x_{3a+2b}(s_{32})$. Then comparing with x, we get

$$t_{01}{}^\theta = t_{10},\ t_{31}{}^\theta + t_{01}{}^\theta t_{10}{}^{3\theta} = t_{11},\ t_{32}{}^\theta + t_{01}{}^{2\theta}t_{10}{}^{3\theta} = t_{21},$$
$$t_{10}{}^{3\theta} = t_{01},\ t_{11}{}^{3\theta} - t_{01}{}^{3\theta}t_{10}{}^{3\theta} = t_{31},\ t_{21}{}^{3\theta} - t_{01}{}^{3\theta}t_{10}{}^{6\theta} = t_{32}.$$

By (5.1), the equations in the first row, above, are equivalent to the equations in the second row. Hence if we set $t_{01} = t$, $t_{31} = u$, $t_{32} = v$, then $x = \alpha(t)\beta(u)\gamma(v)$, where

$$\alpha(t) = x_a(t^\theta)x_b(t)x_{a+b}(t^{\theta+1})x_{2a+b}(t^{2\theta+1}),$$
$$\beta(t) = x_{a+b}(t^\theta)x_{3a+b}(t),$$
$$\gamma(t) = x_{2a+b}(t^\theta)x_{3a+2b}(t).$$

Hence, by [2], Lemma 6, p. 39, we have

(5.5) The group $\mathfrak{U}^1$ of σ-invariant elements in $\mathfrak{U}$ contains $\alpha(t), \beta(t), \gamma(t)$ for all $t \in K$. Every element in $\mathfrak{U}^1$ is written uniquely as $\alpha(t)\beta(u)\gamma(v)$ with $t, u, v \in K$.

By using (4.7) one can show easily

$$\begin{aligned}
&\alpha(t)\alpha(u) = \alpha(t+u)\beta(-tu^{3\theta})\gamma(-t^2u^{3\theta} + tu^{3\theta+1}),\\
&\beta(t)\alpha(u) = \alpha(u)\beta(t)\gamma(-tu),\ \beta(t)\beta(u) = \beta(t+u),\\
&\gamma(t)\alpha(u) = \alpha(u)\gamma(t),\ \gamma(t)\beta(u) = \beta(u)\gamma(t),\\
&\gamma(t)\gamma(u) = \gamma(t+u).
\end{aligned} \tag{5.6}$$

It can be shown easily from the above that the group $\mathfrak{U}^1$ is generated by the elements $\alpha(t)$, $t \in K$.

Before considering σ-invariant elements in $\mathfrak{B}$, we remark that the element ω_0 given in (3.8) can be written

$$\omega_0 = \omega_{a+b}\omega_{3a+b} = \omega_{3a+b}\omega_{a+b} \tag{5.7}$$

and hence is σ-invariant by (3.9) and Theorem 4.8. Now define the elements $\alpha'(t)$, $\beta'(t)$, $\gamma'(t)$ by

$$\alpha'(-t) = \omega_0\alpha(t)\omega_0^{-1}$$
$$= x_{-a}(-t^\theta)x_{-b}(-t)x_{-(a+b)}(-t^{\theta+1})x_{-(2a+b)}(-t^{2\theta+1}),$$
$$\beta'(-t) = \omega_0\beta(t)\omega^{-1} = x_{-(a+b)}(-t^\theta)x_{-(3a+b)}(-t),$$
$$\gamma'(-t) = \omega_0\gamma(t)\omega_0^{-1} = x_{-(2a+b)}(-t^\theta)x_{-(3a+2b)}(-t).$$

Then from (5.5) and the σ-invariance of ω_0 we have

(5.8) The group $\mathfrak{V}^1$ of σ-invariant elements in $\mathfrak{V}$ contains $\alpha'(t), \beta(t), \gamma'(t)$ for all $t \in K$; every element in $\mathfrak{V}^1$ is written uniquely in the form $\alpha'(t)\beta'(u)\gamma'(v)$ with $t, u, v \in K$.

Denote by G^1 the group generated by $\mathfrak{U}^1$ and $\mathfrak{V}^1$. Since

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and since any one of $x_{-(a+b)}(t_1)$, $x_{a+b}(t_2)$ commute with any one of $x_{3a+b}(t_3)$, $x_{-(3a+b)}(t_4)$, we have from (5.7)

(5.9) $$\omega_0 = \beta(1)\beta'(-1)\beta(1) \in G^1.$$

Now we shall consider σ-invariant elements in $\mathfrak{H}$. We have $h(\chi)^\sigma = h(\tilde{\chi})$. Since a and b generate P, χ is determined by $\chi(a)$ and $\chi(b)$. From (4.1) we get

(5.10) $$\tilde{\chi}(a) = \chi(b)^\theta, \quad \tilde{\chi}(b) = \chi(a)^{3\theta}.$$

Hence we have

(5.11) $h(\chi)$ is σ-invariant if and only if $\chi(a) = \chi(b)^\theta$.

By (1.5) and the definition of the elements $\alpha(t), \cdots$, we have

(5.12) If $h(\chi)$ is σ-invariant, then

$$h(\chi)\alpha(t)h(\chi)^{-1} = \alpha(\chi(b)t), \quad h(\chi)\alpha'(t)h(\chi)^{-1} = \alpha'(\chi(b)^{-1}t),$$
$$h(\chi)\beta(t)h(\chi)^{-1} = \beta(\chi(b)^{3\theta+1}t), \quad h(\chi)\beta'(t)h(\chi)^{-1} = \beta'(\chi(b)^{-3\theta-1}t),$$
$$h(\chi)\gamma(t)h(\chi)^{-1} = \gamma(\chi(b)^{3\theta+2}t), \quad h(\chi)\gamma'(t)h(\chi)^{-1} = \gamma'(\chi(b)^{-3\theta-2}t).$$

Denote by $\mathfrak{H}^1$ the group of all σ-invariant elements in $\mathfrak{H}$, and by $\mathfrak{H}^2$ the subgroup of $\mathfrak{H}^1$ consisting of all σ-invariant $h(\chi)$ such that $\chi(b) = z^{3\theta-1}$ with some $z \in K^*$. Note that some elements, for instance -1, in K cannot be written in the form $z^{3\theta-1}$. Since

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z^{-1}-1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ z-1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -z^{-1} \\ 0 & 1 \end{pmatrix}$$

and since any one of $x_{a+b}(t_1)$, $x_{-(a+b)}(t_2)$ commute with any one of $x_{3a+b}(t_3)$, $x_{-(3a+b)}(t_4)$, we have

$$\beta'(z^{-1}-1)\beta(1)\beta'(z-1)\beta(-z^{-1})$$

$$= \phi_{a+b}\begin{pmatrix} z^{\theta} & 0 \\ 0 & z^{-\theta} \end{pmatrix}\phi_{3a+b}\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = h(\chi_{a+b,z^{\theta}})h(\chi_{3a+b,z}).$$

Set

$$\chi = \chi_{a+b,z^{\theta}}\chi_{3a+b,z}.$$

Then it follows from the above that $h(\chi) \in G^1$. We have

$$\chi(b) = z^{b(a+b)\theta+b(3a+b)} = z^{3\theta-1}.$$

Since z is an arbitrary element in K^*, we have

$$\mathfrak{H}^2 \subseteq G^1. \tag{5.13}$$

Now we shall determine all σ-invariant elements in G by (1.13). In view of (3.6), we can assume, for the w_0 given in (2.9), that

$$\omega(w_0) = \omega_0. \tag{5.14}$$

By (1.13) we can write any element x in G in the form $x = uh(\chi)\omega(w)u_1$, where $u \in \mathfrak{U}$ and u_1 is a product of elements of the form $x_r(t)$ with $r > 0$, $w(r) < 0$. We have $x^{\sigma} = u^{\sigma}h(\bar{\chi})\omega(w)^{\sigma}u_1^{\sigma}$. If

$$\omega(w) = \omega_{r_1}\cdots\omega_{r_m}$$

then by Theorem 4.8 we have

$$\omega(w)^{\sigma} = \omega_{\bar{r}_1}\cdots\omega_{\bar{r}_m}$$

and hence by (1.12) and (2.7) we can write $\omega(w)^{\sigma} = h(\chi_1)\omega(\bar{w})$. Then

$$x^{\sigma} = u^{\sigma}h(\bar{\chi}\chi_1)\omega(\bar{w})u_1^{\sigma}. \tag{5.15}$$

Here u_1^{σ} is a product of elements of the form $x_r(t)$, $r > 0$, $w(r) < 0$. We have $\bar{r} > 0$, and $\bar{w}(\bar{r}) = \overline{w(r)} < 0$ by (2.7). Hence the expression for x^{σ} given in (5.15) is exactly in the form specified in (1.13). Assume that x is σ-invariant. Then $x = x^{\sigma}$, and the uniqueness in (1.13) implies

$$u = u^{\sigma}, \quad \chi = \bar{\chi}\chi_1, \quad w = \bar{w}, \quad u_1 = u_1^{\sigma}.$$

Then by (2.9) we have $w = 1$ or $w = w_0$. Then, since we can assume $\omega(1) = 1$, we have from (5.14) $\omega(w)^{\sigma} = \omega(w)$, and hence $h(\chi_1) = 1$, $\chi_1 = 1$. Hence $\chi = \bar{\chi}$ and $h(\chi)$ is σ-invariant. Thus we have proved

(5.16) Any σ-invariant element x in G is written uniquely in the form $x = uh(\chi)\omega u_1$, where: $u \in \mathfrak{U}^1$; $\bar{\chi} = \chi$; $\omega = 1$ or ω_0; $u_1 = 1$ if $\omega = 1$, but u_1 is an arbitrary element in $\mathfrak{U}^1$ if $\omega = \omega_0$.

The form of x given in (5.16), above, will be called the *canonical form* of x.

6. Some identities. We shall compute the canonical forms for $\alpha'(t)$ and $\gamma'(t)$, by following, more or less, the steps (1.15)-(1.17). The results are crucial, as will be seen later, in the discussion of the structure of G^1. Note that our assumption on the field K is the same as in the preceding section.

(6.1) If $t \in K^*$, then

$$\alpha'(t) = \gamma(t^{-(3\theta+2)})\beta(t^{-(3\theta+1)})\alpha(t^{-1})^{-1}h(\chi)\omega_0\alpha(t^{-1})\beta(t^{-(3\theta+1)})\gamma(t^{-(3\theta+2)}),$$

where χ is defined by $\chi(a) = -t^{-2\theta}$, $\chi(b) = -t^{-2}$.

Proof. We have

$$\alpha'(t) = x_{-a}(t^\theta)x_{-b}(t)x_{-(a+b)}(-t^{\theta+1})x_{-(2a+b)}(t^{2\theta+1}).$$

By applying ϕ_{2a+b} to

$$\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} = \begin{pmatrix} 1 & u^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -u^{-1} & 0 \\ 0 & -u \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & u^{-1} \\ 0 & 1 \end{pmatrix} \tag{6.2}$$

we obtain

$$x_{-(2a+b)}(t^{2\theta+1}) = x_{2a+b}(t^{-(2\theta+1)})h(\chi_1)\omega_{2a+b}x_{2a+b}(t^{-(2\theta+1)}),$$

where

$$\chi_1 = \chi_{2a+b,-t^{-(2\theta+1)}}.$$

By transforming the second equation of (4.7) by ω_{a+b}, we obtain

$$x_{-(a+b)}(-t^{\theta+1})x_{2a+b}(t^{-(2\theta+1)}) = x_{2a+b}(t^{-(2\theta+1)})x_a(t^{-\theta})x_{-(a+b)}(-t^{\theta+1}).$$

Hence we have

$$\begin{aligned}\alpha'(t) &= x_{-a}(t^\theta)x_{2a+b}(t^{-(2\theta+1)})x_a(t^{-\theta})x_{-b}(t)x_{-(a+b)}(-t^{\theta+1}) \\ &\quad \cdot h(\chi_1)\omega_{2a+b}x_{2a+b}(t^{-(2\theta+1)}) \\ &= x_{2a+b}(t^{-(2\theta+1)})x_{a+b}(t^{-(\theta+1)})x_{-a}(t^\theta)x_a(t^{-\theta})x_b(t^{-1})h(\chi_2) \\ &\quad \cdot \omega_{-b}x_b(t^{-1})h(\chi_1)\omega_{2a+b}x_{2a+b}(t^{-(2\theta+1)}),\end{aligned}$$

where

$$\chi_2 = \chi_{b,t^{-1}}$$

since, by transforming the second equation of (4.7) by ω_a, we obtain

$$x_{-a}(t^\theta)x_{2a+b}(t^{-(2\theta+1)}) = x_{2a+b}(t^{-(2\theta+1)})x_{a+b}(t^{-(\theta+1)})x_{-a}(t^\theta).$$

Now, by applying ϕ_a to

$$\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\begin{pmatrix} 1 & u^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -u^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix}$$

we obtain

$$x_{-a}(t^{\theta})x_a(t^{-\theta}) = x_a(-t^{-\theta})h(\chi_3)x_{-a}(-t^{\theta}),$$

where

$$\chi_3 = \chi_{a,-1}.$$

Hence we have from the above

$$\begin{aligned}\alpha'(t) &= x_{2a+b}(t^{-(2\theta+1)})x_{a+b}(t^{-(\theta+1)})x_a(-t^{-\theta})h(\chi_3)x_b(t^{-1})x_{-a}(-t^{\theta})\\ &\quad\cdot h(\chi_2)\omega_{-b}x_b(t^{-1})x_{-(a+b)}(-t^{\theta+1})h(\chi_1)\omega_{2a+b}x_{2a+b}(t^{-(2\theta+1)})\\ &= x_{2a+b}(t^{-(2\theta+1)})x_{a+b}(t^{-(\theta+1)})x_a(-t^{-\theta})x_b(-t^{-1})h(\chi_3)\omega_0 x_a(t^{\theta})\\ &\quad\cdot h(\chi_2)^{-1}x_b(t^{-1})x_a(t^{\theta+1})h(\chi_1\circ w_{2a+b})x_{2a+b}(t^{-(2\theta+1)}),\end{aligned}$$

since $\omega_{-b}\omega_{2a+b} = \omega_0$ by (3.8). Since

$$\begin{aligned}&x_a(t^{\theta})h(\chi_2)^{-1}x_b(t^{-1})x_a(t^{\theta+1})h(\chi_1\circ w_{2a+b})\\ &\quad= h(\chi_2)^{-1}h(\chi_1\circ w_{2a+b})x_a(t^{-\theta})x_b(t^{-1})x_{a+b}(-t^{-(\theta+1)})x_{2a+b}(t^{-(2\theta+1)})\\ &\qquad\cdot x_{3a+b}(t^{-(3\theta+1)})x_{3a+2b}(t^{-(3\theta+2)}),\end{aligned}$$

we have

$$\begin{aligned}\alpha'(t) &= x_{2a+b}(t^{-2(\theta+1)})x_{a+b}(t^{-(\theta+1)})x_a(-t^{-\theta})x_b(-t^{-1})h(\chi)\omega_0\\ &\qquad\cdot\alpha(t^{-1})\beta(t^{-(3\theta+1)})\gamma(t^{-(3\theta+2)});\\ \chi &= \chi_3\chi_2(\chi_1\circ w_{2a+b})^{-1}.\end{aligned}$$

From this one can derive (6.1) easily.

(6.3) If $t\in K^*$, then

$$\gamma'(t) = \gamma(t^{-1})\alpha(t^{3\theta-2})^{-1}h(\chi)\omega_0\alpha(-t^{3\theta-2})\gamma(t^{-1}),$$

where χ is defined by $\chi(a) = t^{2-4\theta}$, $\chi(b) = t^{6\theta-4}$.

Proof. We have

$$\gamma'(t) = x_{-(2a+b)}(t^{\theta})x_{-(3a+2b)}(t).$$

From (6.2) we have

$$x_{-(3a+2b)}(t) = x_{3a+2b}(t^{-1})h(\chi_1)\omega_{3a+2b}x_{3a+2b}(t^{-1}),$$

where

$$\chi_1 = \chi_{3a+2b,-t^{-1}}$$

On the other hand, we have by (1.6) and (1.10)

$$x_{-(2a+b)}(t^{\theta}) = \omega_a\omega_b x_{-a}(t^{\theta})\omega_{-b}\omega_{-a}.$$

Hence we have

$$\gamma'(t) = \omega_a\omega_b x_{-a}(t^{\theta})x_{3a+b}(t^{-1})\omega_{-b}\omega_{-a}h(\chi_1)\omega_{3a+2b}x_{3a+2b}(t^{-1}).$$

By transforming the first equation in (4.7) by ω_a, we obtain an expression of the form given in (1.2) for $(x_{-a}(t), x_{3a+b}(u))$. Combining this with (4.7), we get

$$x_{-a}(t^\theta)x_{3a+b}(t^{-1})x_{-a}(-t^\theta)$$
$$= x_{3a+b}(t^{-1})x_{2a+b}(t^{\theta-1})x_{3a+2b}(-t^{3\theta-2})x_{a+b}(-t^{2\theta-1})x_b(t^{3\theta-1}).$$

Hence we have from the above

$$\begin{aligned}\gamma'(t) &= \omega_a\omega_b x_{3a+b}(t^{-1})x_{2a+b}(t^{\theta-1})x_{3a+2b}(-t^{3\theta-2})x_{a+b}(-t^{2\theta-1})\\ &\quad\cdot x_b(t^{3\theta-1})x_{-a}(t^\theta)\omega_{-b}\omega_{-a}h(\chi_1)\omega_{3a+2b}x_{3a+2b}(t^{-1})\\ &= \omega_a x_{3a+2b}(t^{-1})x_{2a+b}(t^{\theta-1})x_{3a+b}(t^{3\theta-2})x_a(t^{2\theta-1})x_{-b}(-t^{3\theta-1})\\ &\quad\cdot x_{-(a+b)}(t^\theta)\omega_{-a}h(\chi_1)\omega_{3a+2b}x_{3a+2b}(t^{-1})\\ &= x_{3a+2b}(t^{-1})x_{a+b}(-t^{\theta-1})x_b(-t^{3\theta-2})x_{-a}(-t^{2\theta-1})x_{-(3a+b)}(-t^{3\theta-1})\\ &\quad\cdot x_{-(2a+b)}(t^\theta)h(\chi_1)\omega_{3a+2b}x_{3a+2b}(t^{-1}).\end{aligned}$$

From (6.2) we have

$$x_{-a}(-t^{2\theta-1}) = x_a(-t^{1-2\theta})h(\chi_2)\omega_{-a}x_a(-t^{1-2\theta}),$$

where

$$\chi_2 = \chi_{a,-t^{1-2\theta}}.$$

Hence

$$\begin{aligned}\gamma'(t) &= x_{3a+2b}(t^{-1})x_{a+b}(-t^{\theta-1})x_b(-t^{3\theta-2})x_a(-t^{1-2\theta})h(\chi_2)\omega_{-a}x_a(-t^{1-2\theta})\\ &\quad\cdot x_{-(3a+b)}(-t^{3\theta-1})x_{-(2a+b)}(t^\theta)h(\chi_1)\omega_{3a+2b}x_{3a+2b}(t^{-1})\\ &= x_{3a+2b}(t^{-1})x_{a+b}(-t^{\theta-1})x_b(-t^{3\theta-2})x_a(-t^{1-2\theta})h(\chi_2)\omega_{-a}\omega_{3a+2b}\\ &\quad\cdot x_a(-t^{1-2\theta})x_b(t^{3\theta-1})x_{a+b}(-t^\theta)h(\chi_1\circ w_{3a+2b})x_{3a+2b}(t^{-1})\\ &= \gamma(t^{-1})\alpha(t^{3\theta-2})^{-1}h(\chi_2)h(\chi_1\circ w_{3a+2b})^{-1}\omega_0\\ &\quad\cdot x_a(-t^{1-2\theta})x_b(-t^{3\theta-2})x_{a+b}(t^{\theta-1})x_{3a+2b}(t^{-1}),\end{aligned}$$

since $\omega_{-a}\omega_{3a+2b} = \omega_0$ by (3.8). Hence we have easily

$$\gamma'(t) = \gamma(t^{-1})\alpha(t^{3\theta-2})^{-1}h(\chi)\omega_0\alpha(-t^{3\theta-2})\gamma(t^{-1}),$$

where

$$\chi = \chi_2(\chi_1\circ w_{3a+2b})^{-1},$$

from which it can be verified easily that $\chi(a) = t^{2-4\theta}$, $\chi(b) = t^{6\theta-4}$. Thus (6.3) is proved.

(6.4) [9] The element $h_0 = h(\chi)$, where $\chi(a) = \chi(b) = -1$, is contained in G^1. h_0 commutes with ω_0, and, for all $t \in K$,

$$h_0\alpha(t)h_0^{-1} = \alpha(-t),\quad h_0\beta(t)h_0^{-1} = \beta(t),\quad h_0\gamma(t)h_0^{-1} = \gamma(-t).$$

[9] The idea of deriving the proposition (6.4) from the canonical form of $\alpha'(t)$ is due to Tits. The author originally proved (6.4) under the assumption that K is finite, using (6.3) and (8.6).

Proof. The first part of (6.4) can be proved by setting $t=1$ in (6.1). The second part is a special case of $\omega_0 h \omega_0^{-1} = h^{-1}$, $h \in \mathfrak{H}$, which follows from (1.7). The third part follows from (5.12).

7. The simplicity of G^1. We shall prove the simplicity of G^1 under the assumption that the field K has more than 3 elements. If K has only 3 elements, then G^1, which is of order $2^3 \cdot 3^3 \cdot 7$, does not seem to be simple, although the author has been unable to verify this; also it is unlikely that any new simple group appears as a composition factor of G^1. In what follows, however, we shall arrange the proof of the simplicity of G^1 in such a way that we use the assumption on K only in the last stage.

Let $H \neq \{1\}$ be a normal subgroup of G^1. Denote, as in Section 5, by $\mathfrak{U}^1$, $\mathfrak{H}^1$, the groups of σ-invariant elements in $\mathfrak{U}$ and $\mathfrak{H}$, respectively.

(7.1) *H contains an element $uh \neq 1$, where $u \in \mathfrak{U}^1$, $h \in \mathfrak{H}^1$.*

Proof. By (5.16) we can assume that H contains $x = uh\omega_0$, where $u \in \mathfrak{U}^1$, $h \in \mathfrak{H}^1$. Let $h_0 \in G^1$ be the element defined in (6.4). Then $h_0 u h_0^{-1} u^{-1} u h \omega_0 \in H$, and hence $y = h_0 u h_0^{-1} u^{-1} \in H$. Since $y \in \mathfrak{U}^1$, we obtain (7.1) if $y \neq 1$. Hence we shall assume $y = 1$. Then from (6.4) and (5.5) we have $u = \beta(t_0)$ with some $t_0 \in K$. Hence $x = \beta(t_0) h \omega_0 \in H$. Let $h = h(\chi)$. Since, by (5.12),

$$\gamma(\chi(b)^{3\theta+2}t)\beta(t_0)h(\chi)\omega_0\gamma(\chi(b)^{3\theta+2}t)^{-1} = \beta(t_0)\gamma(\chi(b)^{3\theta+2}t)\gamma'(t)h(\chi)\omega_0,$$

and since $h(\chi)\omega_0\beta(t_0) \in H$, it folllows that H contains $x_1 = \gamma(\chi(b)^{3\theta+2}t)\gamma'(t)$ for all $t \in K$. By (6.1),

$$x_1 = \gamma(\chi(b)^{3\theta+2}t)\gamma(t^{-1})\alpha(t^{3\theta-2})^{-1}h(\chi_1)\omega_0\alpha(-t^{3\theta-2})\gamma(t^{-1}),$$

where χ_1 is defined by $\chi_1(a) = t^{2-4\theta}$, $\chi_1(b) = t^{6\theta-4}$. Hence H contains

$$\begin{aligned} x_2 &= \alpha(-t^{3\theta-2})\gamma(t^{-1})x_1\gamma(t^{-1})^{-1}\alpha(-t^{3\theta-2})^{-1}x^{-1} \\ &= \alpha(-t^{3\theta-2})\gamma(\chi(b)^{3\theta+2}t - t^{-1})\alpha(t^{3\theta-2})^{-1}h(\chi_1)h(\chi)^{-1}\beta(t_0)^{-1}. \end{aligned}$$

It is clear, in view of (5.6) and (5.12), that $x_2 \in \mathfrak{U}^1\mathfrak{H}^1$ and that $x_2 \neq 1$ if $t \neq 0$. Hence (7.1) is proved.

(7.2) *H contains an element $\neq 1$ in $\mathfrak{U}^1$.*

Proof. By (7.1), H contains $uh \neq 1$, $u \in \mathfrak{U}^1$, $h \in \mathfrak{H}^1$. Assume $h = h(\chi) \neq 1$. Then by (5.11) $\chi(b) \neq 1$. For any $t \in K$, H contains the element

$$\begin{aligned} uh\gamma(t)h^{-1}u^{-1}\gamma(t)^{-1} &= u\gamma(\chi(b)^{3\theta+2}t)u^{-1}\gamma(t)^{-1} \\ &= \gamma((\chi(b)^{3\theta+2}-1)t) \end{aligned}$$

which is $\neq 1$ if $t \neq 0$, since $\chi(b)^{3\theta+2} - 1 = 0$ would imply $\chi(b) = 1$ because of (5.1).

(7.3) H contains an element $\gamma(t)$, $t \neq 0$.

Proof. By (7.2), H contains an element $u \neq 1$ in $\mathfrak{U}^1$.

If $u = \alpha(t_1)\beta(t_2)\gamma(t_3)$ with $t_1 \neq 0$, then for any $t \in K$, H contains, by (5.6),

$$u^{-1}\beta(t)^{-1}u\beta(t) = u^{-1}\alpha(t_1)\beta(t_2)\gamma(t_3 + tt_1) = \gamma(tt_1).$$

If $u = \beta(t_2)\gamma(t_3)$ with $t_2 \neq 0$, then for any $t \in K$, H contains

$$u^{-1}\alpha(t)^{-1}u\alpha(t) = u^{-1}\beta(t_2)\gamma(t_3 - tt_2) = \gamma(-tt_2).$$

In any case, H contains an element $\gamma(t)$ with $t \neq 0$.

(7.4) If H contains $\gamma(t)$, $t \neq 0$, then H contains $\gamma(z^{3\theta+1}t)$ for all $z \in K^*$, and, in particular, $\gamma(t^{-1}) \in H$.

Proof. Let $h(\chi) \in \mathfrak{H}^2$ be such that $\chi(b) = z^{3\theta-1}$. Then $\chi(a) = \chi(b)^\theta = z^{1-\theta}$; $\chi(3a + 2b) = z^{3\theta+1}$. By (5.13), H contains

$$h(\chi)\gamma(t)h(\chi)^{-1} = \gamma(\chi(3a + 2b)t) = \gamma(z^{3\theta+1}t)$$

for any $z \in K^*$. Now set $z = t^{1-3\theta}$. Then $z^{3\theta+1} = t^{-2}$. Hence H contains $\gamma(t^{-1})$.

(7.5) If H contains $\gamma(t)$, $t \neq 0$, then H contains $\alpha(t^{3\theta-2})\beta(t^{1-3\theta})$ and $h(\chi)\omega_0$, where χ is defined by $\chi(a) = t^{2-4\theta}$, $\chi(b) = t^{6\theta-4}$.

Proof. Since G^1 contain ω_0 by (5.9), $\gamma(t) \in H$ implies that $\gamma'(t) \in H$. Then from (6.3) and (7.4) we can conclude that H contains $\alpha(-t^{3\theta-2})\alpha(t^{3\theta-2})^{-1}h(\chi)\omega_0$, where χ is defined by $\chi(a) = t^{2-4\theta}$, $\chi(b) = t^{6\theta-4}$. Since

$$\alpha(u)^{-1} = \alpha(-u)\beta(-u^{3\theta+1})\gamma(u^{3\theta+2})$$

by (5.6), H contains

$$\alpha(-t^{3\theta-2})\alpha(-t^{3\theta-2})\beta(-t^{1-3\theta})\gamma(t^{-1})h(\chi)\omega_0 = \alpha(t^{3\theta-2})\beta(t^{1-3\theta})\gamma(t^{-1})h(\chi)\omega_0.$$

Then by (7.4) H contains $x_1 = \alpha(t^{3\theta-2})\beta(t^{1-3\theta})h(\chi)\omega_0$. Replacing t by $-t$, we see that H also contains $x_2 = \alpha(-t^{3\theta-2})\beta(t^{1-3\theta})h(\chi)\omega_0$. Hence H contains

$$x_2x_1^{-1} = \alpha(-t^{3\theta-2})\alpha(t^{3\theta-2})^{-1} = \alpha(t^{3\theta-2})\beta(t^{1-3\theta})\gamma(t^{-1}).$$

Hence by (7.4) H contains $\alpha(t^{3\theta-2})\beta(t^{1-3\theta})$. Then H contains $h(\chi)\omega_0$, since $x_1 \in H$. Thus (7.5) is proved.

(7.6) H contains ω_0, $\gamma(t)$, $\alpha(t)\beta(t^{3\theta+1})$, and $h(\chi)$ for all $t \in K^*$, where χ is defined by $\chi(a) = t^{2\theta}$, $\chi(b) = t^2$.

Proof. By (7.3) and (7.5), H contains $x = \alpha(t_0^{3\theta-2})\beta(t_0^{1-3\theta})$ for some $t_0 \in K^*$. Hence H contains

$$\beta(u)x\beta(u)^{-1}x^{-1} = \gamma(-ut_0^{3\theta-2}).$$

Since u is an arbitrary element in K, it follows that H contains $\gamma(t)$ for all $t \in K$. Now we can set $t = 1$ in (7.5). Then $h(\chi) = 1$, and hence $\omega_0 \in H$. The rest of (7.6) follows immediately from (7.5) and the fact that $t^{3\theta-2}$ can be an arbitrary element in K^*.

(7.7) H contains the element h_0 defined in (6.4).

Proof. Since H contains ω_0 by (7.6), it follows that H contains

$$\alpha(-t)^{-1}\omega_0\alpha(-t)\omega_0^{-1} = \alpha(-t)^{-1}\alpha'(t),$$

which is, by (6.1), equal to

$$\alpha(-t)^{-1}\gamma(t^{-(3\theta+2)})\beta(t^{-(3\theta+1)})\alpha(t^{-1})^{-1}h(\chi)\omega_0\alpha(t^{-1})\beta(t^{-(3\theta+1)})\gamma(t^{-(3\theta+2)}),$$

where χ is defined by $\chi(a) = -t^{-2\theta}$, $\chi(b) = -t^{-2}$. Hence by (7.6) H contains

$$\alpha(-t)^{-1}\beta(t^{-(3\theta+1)})\alpha(t^{-1})^{-1}h_0.$$

Since H contains $\alpha(-t)^{-1}\beta(t^{3\theta+1})^{-1}$ and $\alpha(t^{-1})\beta(t^{-(3\theta+1)})$ by (7.6), we can conclude that H contains $\beta(t^{3\theta+1})\beta(-t^{-(3\theta+1)})h_0$ for any $t \in K^*$. Setting $t = 1$, we have $h_0 \in H$.

THEOREM 7.8. *If K contains more than 3 elements, then the group G^1 is simple.*

Proof. Let $H \neq \{1\}$ be a normal subgroup of G^1. By (7.6), H contains $h(\chi)$ where χ is defined by $\chi(a) = t^{2\theta}$, $\chi(b) = t^2$. Hence by (5.12), H contains

$$\beta(u)^{-1}h(\chi)\beta(u)h(\chi)^{-1} = \beta((t^{6\theta+2}-1)u)$$

for every $t \neq 0$ and u in K. By (5.1) $t^{6\theta+2}-1 = 0$ would imply $t^4 = 1$. Since K has more than 5 elements, there exists $t \in K^*$ such that $t^4 \neq 1$. Therefore H contains $\beta(t)$ for all $t \in K$. Then from (7.6) it follows that H contains ω_0, $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ for all $t \in K$. Hence $H = G^1$. This proves the simplicity of G^1.

8. Further results on G^1. First we shall consider 2-subgroups of G^1.

(8.1) If $t \in K^*$, then

$$\beta'(t) = \beta(t^{-1})h(\chi)\omega_0\beta(t^{-1}),$$

where χ is defined by $\chi(a) = t^{\theta-1}$, $\chi(b) = t^{1-3\theta}$.

Proof. Applying ϕ_r to (6.2), we obtain

$$x_{-r}(t) = x_r(t^{-1})h(\chi_{r,-t^{-1}})\omega_r x_r(t^{-1}).$$

Since any one of $x_{a+b}(t)$, $x_{-(a+b)}(t)$ commute with any one of $x_{3a+b}(u)$, $x_{-(3a+b)}(u)$, and since $\omega_{a+b}\omega_{3a+b} = \omega_0$ by (3.8), we have from the above,

$$\beta'(t) = x_{-(a+b)}(t^\theta)x_{-(3a+b)}(t) = \beta(t^{-1})h(\chi)\omega_0\beta(t^{-1}),$$

where

$$\chi = \chi_{a+b,-t^{-\theta}}\chi_{3a+b,-t^{-1}}.$$

It can be verified easily that $\chi(a) = t^{\theta-1}$, $\chi(b) = t^{1-3\theta}$. Thus (8.1) is proved.

(8.2) The element $x_0 = \beta(1)\omega_0\beta(-1)$ commutes with ω_0.

Proof. From (8.1) we have

$$\beta'(1) = \beta(1)\omega_0\beta(1), \qquad \beta'(-1) = \beta(-1)\omega_0\beta(-1).$$

Hence $x_0 = \beta'(1)\beta(1) = \beta(-1)\beta'(-1)$, from which it follows immediately that $\omega_0 x_0 \omega_0^{-1} = x_0$.

THEOREM 8.3. *The group G^1 contains an elementary abelian 2-subgroup of order* 8.

Proof. Let x_0 and h_0 be the elements defined in (8.2) and (6.4), respectively. Then, by (8.2) and (6.4), it is clear that the 3 elements x_0, h_0, ω_0 of order 2 commute with each other. By (5.16), h_0 and ω_0 generate a subgroup of order 4 which does not contain x_0. Hence x_0, h_0, and ω_0 generate an elementary abelian 2-group of order 8.

Now assume that K is a finite field of $q = 3^{2n+1}$ elements, where $n = 0, 1, 2, \cdots$. Then the automorphism θ of K defined by $t^\theta = t^m$, where $m = 3^n$, satifies $3\theta^2 = 1$. Hence the group G^1 can be defined. Since K^* is a cyclic group of order $q-1$, the subgroup K^{*2} of square elements is of index 2 in K^*. By (5.4), $-1 \notin K^2$. Hence K^* is generated by K^{*2} and -1. Hence by (6.1), (6.4), and (5.16) we have

(8.4) If K is finite, then G^1 is the group of all σ-invariant elements in G.

By (5.16) and (8.4), above, the order of G^1 is

$$q^3(q-1)+q^3(q-1)=q^3(q-1)(q^3+1).$$

Since $q-1\equiv 2 \pmod 8$, $q^3+1\equiv 4 \pmod 8$, the above order is divisible by 8 but not by 16. The 2-subgroup of order 8 given in (8.3) is then a Sylow subgroup. Hence

THEOREM 8.5. *If K is a finite field of $q=3^{2n+1}$ elements, where $n=0,1,2,\cdots$, then the order of G^1 is $q^3(q-1)(q^3+1)$, and a Sylow 2-subgroup of G^1 is an elementary abelian 2-group of order 8.*

All the previously known finite simple groups whose orders contain the prime 2 with the exact exponent 3 are: $L_3(2)$, $L_2(q)$ with $q\equiv \pm 7 \pmod{16}$, and the alternating group of degree 7 (cf. [1]). Among these only the first group, of order 168, has an elementary abelian Sylow 2-subgroup, and for the rest the Sylow 2-subgroups are dihedral. Hence Theorem 8.5, above, shows that the finite simple groups of the form G^1 are new. In passing, we note that all the known finite simple groups whose orders contain the prime 2 with the exact exponent 2 are the groups $L_2(q)$ with $q\equiv \pm 3 \pmod 8$, $q>3$ (cf. [1]). The Sylow 2-subgroups of these groups are, of course, elementary abelian, since they cannot be cyclic.

We shall devote the rest of this section to a study of the group S of all $z\in K^*$ such that $\chi(b)=z$ for some $h(\chi)\in G^1$. If K is finite, then (8.4) shows that $S=K^*$. In the general case, we already know that $z^{3\theta-1}\in S$ for any $z\in K^*$ by (5.13), and that $-1\in S$ by (6.4). Except for these, however, we know very little about S. We shall prove

THEOREM 8.6. *We have $1+z^2\in S$ for any $z\in K$.*

First we shall prove a few lemmas.

(8.7) Let

$$x=x_{3a+2b}(t_{32})x_{3a+b}(t_{31})x_{2a+b}(t_{21})x_{a+b}(t_{11})x_b(t_{01})x_a(t_{10})\omega_0$$

with $t_{ij}\in K$. If

$$t_{10}\neq 0,\ 1-t_{31}\neq 0,\ 1+t_{10}t_{21}-t_{31}\neq 0,$$

then

$$x_{-(3a+b)}(-1)x=x_{3a+2b}(u_{32})x_{3a+b}(u_{31})x_{2a+b}(u_{21})x_{a+b}(u_{11})\cdot x_b(u_{01})x_a(u_{10})h(\chi)\omega_0 x',$$

where $x'\in \mathfrak{U}$, where χ is defined by

$$\chi(a)=(1+t_{10}t_{21}-t_{31})^{-2}(1-t_{31}),$$
$$\chi(b)=(1+t_{10}t_{21}-t_{31})^{3}(1-t_{31})^{-2},$$

and where

$$u_{11} = t_{11} + t_{10}^{-2}t_{31} - (t_{21} - t_{10}^{-1}t_{31})^2 - (t_{10}^{-1}t_{31} - t_{21} - t_{10}^{-1})^2 t_{31}(1 - t_{31})^{-1},$$
$$u_{10} = -(t_{10}^{-1}t_{31} - t_{21} - t_{10}^{-1})^{-1},$$
$$u_{01} = t_{32} - t_{10}^{-3}t_{31}^2 - t_{01}t_{31} + t_{01} - t_{10}^{-3}t_{31} + (t_{21} - t_{10}^{-1}t_{31})^3 - (t_{10}^{-1}t_{31} - t_{21} - t_{10}^{-1})^3 t_{31}(1 - t_{31})^{-1}.$$
$$u_{32} = t_{32} - t_{10}^{-3}t_{31}^2 - t_{01}t_{31} - (t_{21} - t_{10}^{-1}t_{31})^3 - (t_{10}^{-1}t_{31} - t_{21} - t_{10}^{-1})^3 t_{31}^2(1 - t_{31})^{-2}.$$

Proof. We merely follow the steps (1.15)-(1.17) mechanically to find the canonical form of the element $x_{-(3a+b)}(-1)x = \omega_a x_{-b}(-1)\omega_a^{-1}x$. We omit the details.

(8.8) Let

$$y = x_{3a+2b}(u_{32})x_{3a+b}(u_{31})x_{2a+b}(u_{21})x_{a+b}(u_{11})x_b(u_{01})x_a(u_{10})\omega_0$$

with $u_{ij} \in K$. If

$$u_{01} \neq 0, 1 - u_{11} - u_{01}u_{10} \neq 0, 1 + u_{01}u_{32} - u_{11}^3 \neq 0,$$

then we have

$$x_{-(a+b)}(-1)y = y''h(\chi_1)\omega_0 y',$$

where y' and y'' are in $\mathfrak{U}$, and where χ_1 is given by

$$\chi_1(a) = (1 + u_{01}u_{32} - u_{11}^3)(1 - u_{11} - u_{01}u_{10})^{-2},$$
$$\chi_1(b) = (1 + u_{01}u_{32} - u_{11}^3)^{-2}(1 - u_{11} - u_{01}u_{10})^3.$$

Proof. We make use of the automorphism σ of G. Set

$$x = y^\sigma = x_{3a+2b}(t_{32})x_{3a+b}(t_{31}) \cdots x_a(t_{10})\omega_0.$$

Then by an easy computation we get

$$t_{10} = u_{01}^\theta, \quad t_{31} = u_{11}^{3\theta} + u_{01}^{3\theta}u_{10}^{3\theta}, \quad t_{21} = u_{32}^\theta + u_{01}^{2\theta}u_{10}^{3\theta}.$$

Hence $t_{10} \neq 0$, $1 - t_{31} \neq 0$ are clear. Also

$$t_{31} - t_{10}t_{21} = u_{11}^{3\theta} - u_{01}^\theta u_{32}^\theta = (u_{11}^3 - u_{01}u_{32})^\theta \neq 1.$$

Hence we can apply (8.7) to x. Set

$$x_{-(3a+b)}(-1)x = x''h(\chi)\omega_0 x',$$

where x', x'' are in $\mathfrak{U}$ and where χ is given in (8.7). Applying σ to the above, we get

$$x_{-(a+b)}(-1)y = y''h(\bar{\chi})\omega_0 y',$$

where $y'' = (x'')^\sigma$, $y' = (x')^\sigma$. Hence we can set $\chi_1 = \chi$. We shall compute the values of χ_1. By (8.7) we have

$$\begin{aligned}\chi(a) &= (1 + t_{10}t_{21} - t_{31})^{-2}(1 - t_{31}) \\ &= (1 + u_{01}{}^{\theta}u_{32}{}^{\theta} - u_{11}{}^{3\theta})^{-2}(1 - u_{11}{}^{3\theta} - u_{01}{}^{3\theta}u_{10}{}^{3\theta}).\end{aligned}$$

Since $\chi_1(b) = \chi(a)^{3\theta}$ by (5.10), we have

$$\chi_1(b) = (1 + u_{01}u_{32} - u_{11}{}^3)^{-2}(1 - u_{11} - u_{01}u_{10})^3.$$

Similarly we can compute $\chi_1(a)$. Thus (8.8) is proved.

Proof of Theorem 8.6. Set $x = \gamma(-t^{3\theta+2})\alpha(-t)^{-1}\omega_0$ with $t \neq 0, \pm 1$. Then

$$x = x_{3a+2b}(-t^{3\theta+2})x_{a+b}(-t^{\theta+1})x_b(t)x_a(t^\theta)\omega_0$$

Since $t \neq 0$, we can apply (8.7) to x. We obtain

$$y = x_{-(3a+b)}(-1)x = x_{3a+2b}(u_{32}) \cdots x_b(u_{01})x_a(u_{10})\omega_0 x',$$

where $x' \in \mathfrak{U}$, and

$$u_{11} = -t^{\theta+1}, \quad u_{01} = -t^{3\theta+2} + t, \quad u_{32} = -t^{3\theta+2}, \quad l_{01} = t^\theta.$$

We wish to apply (8.8) to y. The condition on u_{ij} in (8.8) are easily found to be equivalent to $t^2 \neq \pm 1$, and hence by (5.4), to $t \neq \pm 1$. Since these are assumed, we can apply (8.8) to y, and obtain

$$\beta'(-1)x = x_{-(a+b)}(-1)y = y''h(\chi_1)\omega_0 y',$$

where y', y'' are in $\mathfrak{U}^1$ (since x and $\beta'(-1)$ are σ-invariant), and where $h(\chi_1)$, which is also σ-invariant, is given by

$$\chi_1(b) = (1 + t^{6\theta+4})^{-2}(1 + t^{4\theta+2})^3.$$

Since x is in G^1, it is clear from the above that $h(\chi_1)$ is in G^1. By (6.1) and (6.4), $K^{*2} \subseteq S$. Then the above implies that $1 + t^{4\theta+3} \in S$ whenever $t \neq 0, \pm 1$. Write $z = t^{2\theta+1}$. Then $t = z^{6\theta-3}$, and the conditions $t \neq 0, \pm 1$ are easily seen to be equivalent to $z \neq 0, \pm 1$. If $z = 0$ or ± 1, then $1 + z^2 = \pm 1$, and hence by (6.4) $1 + z^2 \in S$. Thus (8.6) is proved.

9. Automorphisms of G^1. If $t \to t'$ is an automorphism of the field K which commutes with θ, then, since G^1 is generated by $\alpha(t), \cdots, \gamma'(t)$, it can be seen easily that the map $\alpha(t) \to \alpha(t'), \cdots, \gamma'(t) \to \gamma'(t')$ can be extended to an automorphism ρ of G^1. We shall call ρ the automorphism of G^1 induced by the automorphism $t \to t'$ of K. If K is a finite field, then any

automorphism of K commutes with θ and hence induces an automorphism of G^1. We shall prove

THEOREM 9.1. *If K is a finite*[10] *field possessing more than 3 elements, then every automorphism of G^1 can be written uniquely in the form $\rho_1\rho_2$, were ρ_1 is an automorphism of G^1 induced by an automorphism of K and where ρ_2 is an inner automorphism of G^1.*

Proof. Let $x \to x^\rho$ be an automorphism of G^1. Since $\mathfrak{U}^1$ is clearly a Sylow 3-subgroup of G^1, and since any two Sylow 3-subgroups are conjugate, it follows that $\mathfrak{U}^1$ and $(\mathfrak{U}^1)^\rho$ are conjugate; there exists an inner automorphism ρ' such that $\rho\rho'$ maps $\mathfrak{U}^1$ onto itself. Hence we can assume from the beginning that

$$(\mathfrak{U}^1)^\rho = \mathfrak{U}^1. \tag{9.2}$$

By (5.6) it can be seen easily that the center of $\mathfrak{U}^1$ consists of the elements $\gamma(t)$, $t \in K$. Hence by (9.2) there exists an automorphism $t \to t'$ of the additive group of K such that $\gamma(t)^\rho = \gamma(t')$. Write $c = 1' \neq 0$. Since K is assumed to be finite, there exists, by (8.4), $h = h(\chi) \in G^1$ such that $\chi(b) = c^{3\theta-2}$. Denote by ρ' the inner automorphism defined by $x^{\rho'} = hxh^{-1}$. Then $\gamma(t)^{\rho\rho'} = h\gamma(t')h^{-1} = \gamma(\chi(b)^{3\theta+2}t') = \gamma(c^{-1}t')$, and hence $\gamma(1)^{\rho\rho'} = \gamma(1)$. Clearly ρ' maps $\mathfrak{U}^1$ onto itself. Hence we can assume from the beginning that ρ satisfies not only (9.2) but also

$$\gamma(t)^\rho = \gamma(t'), \tag{9.3}$$

where $t \to t'$ is an automorphism of the additive group of K such that $1' = 1$.

We shall show that the group $\mathfrak{U}^1\mathfrak{H}^1$ is the normalizer of $\mathfrak{U}^1$ in G^1. It is clear that $\mathfrak{U}^1\mathfrak{H}^1$ normalizes $\mathfrak{U}^1$. Suppose $x \in G^1$ normalizes $\mathfrak{U}^1$, and let $x = uh\omega u_1$ be the canonical form of x given in (5.16). Then ω normalizes $\mathfrak{U}^1$. Hence $\omega \neq \omega_0$, and we have $\omega = 1$. Thus $x \in \mathfrak{U}^1\mathfrak{H}^1$. Then by (9.2) we get

$$(\mathfrak{U}^1\mathfrak{H}^1)^\rho = \mathfrak{U}^1\mathfrak{H}^1, \tag{9.4}$$

from which it follows that there exists a homomorphism $h(\chi) \to h(\chi')$ of H^1 into itself such that

$$h(\chi)^\rho h(\chi')^{-1} \in \mathfrak{U}^1. \tag{9.5}$$

By (9.3) and (9.5) we obtain

$$\chi'(b)^{3\theta+2}t' = (\chi(b)^{3\theta+2}t)'.$$

[10] See footnote 3.

Setting $t=1$ in the above, and using $1'=1$ we obtain

$$\chi'(b)^{3\theta+2}=(\chi(b)^{3\theta+2})'. \tag{9.6}$$

Then from the above we get

$$(\chi(b)^{3\theta+2})'t'=(\chi(b)^{3\theta+2}t)'.$$

Since, by (8.4) and (5.3), every element in K^* can be written in the form $\chi(b)^{3\theta+2}$ with $h(\chi)\in G^1$, it follows that $u't'=(ut)'$ for any u, t in K. Since $t\to t'$ is an automorphism of the additive group of K, we have thus shown that the map $t\to t'$ is an automorphism of K. Denote by ρ' the automorphism of G^1 induced by the inverse of the automorphism $t\to t'$. Then $\gamma(t')^{\rho'\rho}=\gamma(t)^{\rho}=\gamma(t')$ for all $t\in K$. Also, it is clear that ρ' maps $\mathfrak{U}^1$ into itself. Hence we can assume, for the proof of Thseorem 9.1, that $t'=t$ for all $t\in K$. Then

$$\gamma(t)^{\rho}=\gamma(t) \tag{9.7}$$

for all $t\in K$, and, by (9.6), $\chi'=\chi$. Hence, by (9.5),

$$h(\chi)^{\rho}h(\chi)^{-1}\in\mathfrak{U}^1. \tag{9.8}$$

Now, by (5.16) one can show easily that $\omega_0\mathfrak{U}^1\omega_0^{-1}\cap\mathfrak{H}^1=\{1\}$, which, combined with (9.4), shows that $\omega_0{}^{\rho}$ must be of the form $\omega_0{}^{\rho}=uh\omega_0u'$, where u, $u'\in\mathfrak{U}^1$, $h\in\mathfrak{H}^1$. Then $\omega_0{}^2=1$ implies $uh\omega_0u'uh\omega_0u'=1$, which in turn implies $u'u=1$. Denote by ρ' the inner automorphism $x\to u^{-1}xu$ of G^1. Then $\omega_0{}^{\rho\rho'}=h\omega_0$, and we can replace ρ by $\rho\rho'$ in (9.2), (9.7), and (9.8) without losing the validity of the statements. Hence we can assume

$$\omega_0{}^{\rho}=h(\chi_0)\omega_0. \tag{9.9}$$

We shall show that $\rho=1$ follows from (9.2), (9.7), (9.8), and (9.9). This is clearly sufficient for the proof of Theorem 9.1. By (9.8) we can write $h(\chi)^{\rho}=u_{\chi}h(\chi)$ with $u_{\chi}\in\mathfrak{U}^1$. Substituting this and (9.9) to $\omega_0{}^{-1}h(\chi)\omega_0=h(\chi)^{-1}$ one can derive easily $u_{\chi}=1$. Thus we have, for all $h(\chi)\in\mathfrak{H}^1$,

$$h(\chi)^{\rho}=h(\chi). \tag{9.10}$$

By (9.2) we can write $\alpha(1)^{\rho}=\alpha(c)\beta(c_1)\gamma(c_2)$ with c, c_1, and c_2 in K. Then by (9.10), (5.12), and the fact that $\chi(b)$ can be regarded as an arbitrary element in K^*, we have

$$\alpha(t)^{\rho}=\alpha(ct)\beta(c_1t^{3\theta+1})\gamma(c_2t^{3\theta+2}) \tag{9.11}$$

for all $t\in K$.

Let $\mathfrak{Z}^1$ be the center of $\mathfrak{U}^1$, and $\mathfrak{Z}^2/\mathfrak{Z}^1$ the center of $\mathfrak{U}^1/\mathfrak{Z}^1$. Then by (5.6) one can see easily that $\mathfrak{Z}^2$ consists of the elements $\beta(t)\gamma(u), t, u \in K$. Hence $\beta(t)^\rho = \beta(t_1)\gamma(t_2)$. Let h_0 be the element defined in (6.4). Then by (9.10), $h_0{}^\rho = h_0$. Hence we have $\beta(t)^\rho = \beta(t_1)\gamma(-t_2)$. Then $t_2 = -t_2$, $t_2 = 0$. Thus we have $\beta(t)^\rho = \beta(t_1)$. Combining this and (9.11) with the second identity of (5.6) one can deduce easily $cut_1 = ut$ for all u, t in K. Hence $c \neq 0$, and $t_1 = c^{-1}t$. Thus we have proved that

$$\beta(t)^\rho = \beta(c^{-1}t) \tag{9.12}$$

holds for all $t \in K$. Now combining (9.11) and (9.12) with the first identity of (5.6) one can deduce

$$c_1(t+u)^{3\theta+1} - c^{-1}tu^{3\theta} = c_1t^{3\theta+1} + c_1u^{3\theta+1} - ct(cu)^{3\theta}$$

for all $t, u \in K$. Since

$$(t+u)^{3\theta+1} = t^{3\theta+1} + t^{3\theta}u + tu^{3\theta} + u^{3\theta+1}$$

we have $c_1t^{3\theta}u = (c^{-1} - c^{3\theta+1} - c_1)tu^{3\theta}$ for all $t, u \in K$. Now assume that K has more than 3 elements. Then we have $c_1 = 0$, $c^{-1} - c^{3\theta+1} - c_1 = 0$. Hence $c^{3\theta+2} = 1$, $c = 1$. Thus we have

$$\alpha(t)^\rho = \alpha(t)\gamma(c_2t^{3\theta+2}), \qquad \beta(t)^\rho = \beta(t)$$

for all $t \in K$. Combining these again with the first identity of (5.6), we can deduce

$$c_2(t+u)^{3\theta+2} = c_2t^{3\theta+2} + c_2u^{3\theta+2}$$

for all $t, u \in K$. Since

$$(t+u)^{3\theta+2} = t^{3\theta+2} + u^{3\theta+2} - t^{3\theta}(tu - u^2) + u^{3\theta}(t^2 - tu)$$

we have, by setting $u = 1$, $c_2(t^{3\theta} - t)(t-1) = 0$ for all $t \in K$. It can be easily seen that $t^{3\theta} - t = 0$ implies that $t = 0$ or ± 1. Since K is assumed to have more than 3 elements, we conclude $c_2 = 0$. Thus we have proved, for all $t \in K$,

$$\alpha(t)^\rho = \alpha(t), \qquad \beta(t)^\rho = \beta(t). \tag{9.13}$$

Now, by (8.1), (9.7), (9.9), (9.10), and (9.13), we have

$$h(\chi_0)\omega_0\beta(-1)\omega_0{}^{-1}h(\chi_0)^{-1} = \beta(1)h(\chi_0)\omega_0\beta(1).$$

Then by (5.12) we have

$$\beta'(\chi_0(b)^{-3\theta-1}) = \beta(1)h(\chi_0)\beta(1)^{-1}\beta'(1).$$

Then $\omega_0 \mathfrak{U}^1 \omega_0^{-1} \cap \mathfrak{U}^1 = \{1\}$ implies $\beta(1)h(\chi_0)\beta(1)^{-1} = 1$, $h(\chi_0) = 1$. Thus we have proved $\omega_0^{\rho} = \omega_0$. Then by (9.7) and (9.13) we have $\rho = 1$, since G^1 is generated by the elements $\alpha(t)$, $\beta(t)$, $\gamma(t)$, and ω_0. Thus the proof of Theorem 9.1 is complete, except for the uniqueness part.

To prove the uniqueness, let $t \to t'$ be an automorphism of K, and suppose that the automorphism of G^1 induced by it is the inner automorphism $x \to axa^{-1}$. Then clearly $a\mathfrak{U}^1 a^{-1} = \mathfrak{U}^1$, and, since the normalizer of $\mathfrak{U}^1$ on G^1 is $\mathfrak{U}^1\mathfrak{H}^1$, a must be of the form $a = uh$, where $u \in \mathfrak{U}^1$, $h \in \mathfrak{H}^1$. On the other hand, since $1' = 1$, $(-1)' = -1$, a must commute with $\omega_0 = \beta(1)\beta'(-1)\beta(1)$. From this we can conclude easily $u = 1$, $h = 1$, and that $t \to t'$ is the identity automorphism of K. Thus the uniqueness part is also proved.

UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, B. C.

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ALGEBRAIC ISOMORPHISMS BETWEEN ORDERED BASES.*

By J. R. RINGROSE.

1. Introduction. Let H be a complex Hilbert space, and denote by (H,H) the class of all bounded linear operators acting in H. An *operator algebra over* H is a subset $\mathcal{J}$ of (H,H) such that, if $A,B\in\mathcal{J}$ and λ is a complex number, then $A+B, AB, \lambda A\in\mathcal{J}$. If $\mathcal{J}_1$, $\mathcal{J}_2$ are operator algebras over Hilbert spaces H_1, H_2 respectively, an *algebraic isomorphism* from $\mathcal{J}_1$ onto $\mathcal{J}_2$ is a one to one linear and multiplicative mapping ϕ from $\mathcal{J}_1$ onto $\mathcal{J}_2$. If there is a non-singular continuous linear transformation Q from H_2 onto H_1 such that

$$\phi(T)=Q^{-1}TQ \qquad (T\in\mathcal{J}_1),$$

we say that ϕ is a *spatial isomorphism,* and is *implemented by* Q. The main object of the present paper is to prove that, if $\mathcal{J}_1$ and $\mathcal{J}_2$ are ordered bases (in the sense of [3]), then every algebraic isomorphism from $\mathcal{J}_1$ onto $\mathcal{J}_2$ is spatial.

2. Notation and definitions. Let $\mathcal{J}$ be an operator algebra over a Hilbert space H. We denote by $\mathcal{J}^*$ the algebra $\{T^*: T\in\mathcal{J}\}$. By a *hull* of $\mathcal{J}$ we shall mean a closed subspace L of H which is invariant under each operator in $\mathcal{J}$. A *hull idempotent* of $\mathcal{J}$ is an idempotent in (H,H) whose range is a hull of $\mathcal{J}$. A self-adjoint hull idempotent will be called a *hull projection. A full class of hull idempotents of* $\mathcal{J}$ is a family $\mathcal{F}$ of hull idempotents such that, given any hull L of $\mathcal{J}$, there is a member of $\mathcal{F}$ whose range is L.

Following Kadison and Singer [3], we shall say that the operator algebra algebra $\mathcal{J}$ is a *maximal triangular algebra* if

(i) $\mathcal{A}=\mathcal{J}\cap\mathcal{J}^*$ is a maximal abelian sub-algebra of (H,H);

(ii) amongst algebras satisfying (i), $\mathcal{J}$ is maximal.

Under these conditions, $\mathcal{A}$ is a commutative von Neumann algebra, called the *diagonal* of $\mathcal{J}$. It is shown in [3] that the hulls of a maximal triangular algebra are linearly ordered (by inclusion), and that each hull projection lies

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in the diagonal of the algebra. If the hull projections of a maximal triangular algebra $\mathcal{T}$ generate the diagonal (as a von Neumann algebra), we shall call $\mathcal{T}$ an *ordered basis*. The motivation for these definitions is indicated in [3]. We note that, owing to the need in this paper to consider non-orthogonal projections, our use of the term *hull* differs slightly from that of [3].

We now give an example which is of some importance in the theory. Let ρ be a non-negative measure defined on the Borel subsets of the closed interval $[0,1]$, and denote by H_ρ the space $L^2([0,1]:\rho)$ of ρ-measurable functions whose squares are ρ-summable over $[0,1]$. Given a ρ-essentially bounded function $f(t)$, let M_f be the operator of multiplication by f. The set

$$\mathcal{A}_\rho = \{M_f : f \in L^\infty([0,1]:\rho)\}$$

is a maximal abelian sub-algebra of (H_ρ, H_ρ), the *multiplication algebra* over H_ρ. This algebra contains the orthogonal projection $E(\tau)$ of multiplication by the characteristic function of τ, whenever τ is a Borel subset of $[0,1]$. We refer to $E(\tau)$ as the *spectral measure in* $\mathcal{A}_\rho$. We define $E_\lambda = E([0,\lambda])$, $E_{\lambda-} = E([0,\lambda))$ $(0 \leqq \lambda \leqq 1)$. Now let $\mathcal{T}_\rho$ be the class of all bounded linear operators acting in H_ρ which leave invariant the range of each E_λ. Then $\mathcal{T}_\rho$ is an ordered basis over H_ρ, with diagonal $\mathcal{A}_\rho$ and hull projections E_λ, $E_{\lambda-}$ $(0 \leqq \lambda \leqq 1)$. For these facts we refer to [3]. The importance of this example lies in the following representation theorem, which is proved in [3] (see p. 248).

THEOREM. *Let $\mathcal{T}$ be an ordered basis over a separable Hilbert space H. Then it is possible to choose a non-negative measure ρ defined on the Borel subsets of the interval* [0,1], *and a unitary operator U from H onto H_ρ, in such a way that $\mathcal{T} = U^{-1}\mathcal{T}_\rho U$.*

As an immediate consequence of this theorem, we note that an ordered basis consists exactly of those operators which leave invariant each of its hulls.

If $\mathcal{T}$ is an operator algebra over a Hilbert space H, we denote by $\mathcal{T}'$ the commutant of $\mathcal{T}$: that is,

$$\mathcal{T}' = \{S \in (H,H) : ST = TS \ (T \in \mathcal{T})\}.$$

The von Neumann algebras over H can be defined as those for which $\mathcal{T} = \mathcal{T}^* = (\mathcal{T}')'$.

3. Some preliminary lemmas.

LEMMA 1. *The following properties of elements of an operator algebra*

$\mathfrak{J}$ over a Hilbert space H are conserved by any algebraic isomorphism between two operator algebras.

(i) *E is idempotent.*

(ii) *E, E' are idempotents and $E(H) \subset E'(H)$.*

(iii) *E, E' are idempotents and $E(H) = E'(H)$.*

(iv) *E is a hull idempotent of $\mathfrak{J}$.*

Proof. In order to prove parts (i), (ii) and (iv) we shall state in each case an equivalent property which is obviously preserved by algebraic isomorphism. Part (iii) is immediate from (ii).

(i) $E^2 = E$.

(ii) $E^2 = E'E = E$, $(E')^2 = E'$.

(iv) $E^2 = E$, $TE = ETE$ $(T \in \mathfrak{J})$.

LEMMA 2. *Let $\mathfrak{J}_1$, $\mathfrak{J}_2$ be maximal triangular algebras, and let ϕ be an algebraic isomorphism from $\mathfrak{J}_1$ onto $\mathfrak{J}_2$. Then if $\mathfrak{F}$ $(\subset \mathfrak{J}_1)$ is a full set of hull idempotents for $\mathfrak{J}_1$, the family $\phi(\mathfrak{F}) = \{\phi(E) : E \in \mathfrak{F}\}$ is a full set of hull idempotents for $\mathfrak{J}_2$.*

Proof. It is immediate from Lemma 1(iv) that $\phi(\mathfrak{F})$ is a family of hull idempotents for $\mathfrak{J}_2$. Suppose now that L is any hull of $\mathfrak{J}_2$, and let F be the corresponding hull projection. Since $\mathfrak{J}_2$ is a maximal triangular algebra, $F \in \mathfrak{J}_2$. Thus we may set $F = \phi(E)$, where $E \in \mathfrak{J}_1$ and (by Lemma 1(iv)) is a hull idempotent of $\mathfrak{J}_1$. Since $\mathfrak{F}$ is a full set of hull idempotents for $\mathfrak{J}_1$, there is an element E' in $\mathfrak{F}$ having the same range as E. Let $\phi(E') = F'$. Then $F' \in \phi(\mathfrak{F})$, and has the same range as F (by Lemma 1(iii)). Since the range of F is L, it follows that $\phi(\mathfrak{F})$ is a full set of hull idempotents for $\mathfrak{J}_2$.

LEMMA 3. *Let e, f be elements of the Hilbert space H_ρ, and let S be the bounded linear operator acting in H_ρ which is defined by the equation $Sx = \langle x, e\rangle f$ $(x \in H_\rho)$. Then a necessary and sufficient condition that S should be a member of the ordered basis $\mathfrak{J}_\rho$ is that there exist a $\mu \in [0,1]$ such that $E_{\mu-}e = 0$, $E_\mu f = f$. The notations are those introduced in § 2.*

Proof. Suppose that $S \in \mathfrak{J}_\rho$, and let $\mu = \inf\{\lambda : E_\lambda f = f\}$. The right continuity of E_λ implies that $E_\mu f = f$, while from the definition of μ it follows that $E_\lambda f \neq f$ $(\lambda < \mu)$. Now

$$SE_\lambda e = \| E_\lambda e \|^2 f,$$
$$E_\lambda S E_\lambda e = \| E_\lambda e \|^2 E_\lambda f.$$

Since $SE_\lambda = E_\lambda S E_\lambda$, while $f \neq E_\lambda f$ when $\lambda < \mu$, we deduce that for such λ, $E_\lambda e = 0$. When $\lambda \nearrow \mu$ we obtain $E_{\mu-} e = 0$.

Now suppose conversely that for some $\mu \in [0,1]$ we have $E_{\mu-} e = 0$, $E_\mu f = f$. If $\lambda < \mu$, then

$$SE_\lambda x = \langle E_\lambda x, e\rangle f = \langle x, E_\lambda e\rangle f = 0 \qquad (x \in H_\rho),$$

and thus $SE_\lambda = E_\lambda S E_\lambda$ $(\lambda < \mu)$. On the other hand, the last equation is satisfied when $\lambda \geqq \mu$, since then $f = E_\lambda f$. Hence S leaves invariant the range of each E_λ, and is therefore a member of $\mathfrak{T}_\rho$. This completes the proof of the lemma.

LEMMA 4. *Let $\mathfrak{T}$ be an ordered basis over a separable Hilbert space H, and suppose that $A \in \mathfrak{T}$. Then A has rank unity if and only if* (i) *$A \neq 0$, and* (ii) *if $B, C \in \mathfrak{T}$ and $BAC = 0$, then at least one of BA, AC is zero. Consequently, the property of having rank unity is preserved by algebraic isomorphism between ordered bases.*

Proof. We first suppose that A has rank unity. Then we may choose e, f $(\in H)$ in such a way that $Ax = \langle x, e\rangle f$ $(x \in H)$. We then have

$$BAx = \langle x, e\rangle Bf, \quad ACx = \langle x, C^*e\rangle f, \quad BACx = \langle x, C^*e\rangle Bf \qquad (x \in H).$$

If $BAC = 0$, one at least of C^*e, Bf is zero; hence one at least of BA, AC is zero. Since $A \neq 0$, conditions (i) and (ii) are satisfied.

Now suppose that A has rank exceeding unity. Because of the representation theorem for ordered bases which is stated in § 2, it is sufficient to consider the case in which $\mathfrak{T} = \mathfrak{T}_\rho$. We shall use the notations introduced in § 2, and write L_λ for the range of the projection E_λ. Let

$$\alpha = \sup\{\lambda : E_\lambda = 0\}, \qquad \beta = \inf\{\lambda : E_\lambda = I\}.$$

Thus $E_{\alpha-} = 0$, $E_\beta = I$. We now consider separately four possibilities, enumerated below as cases 1 and 2, in each of which there are alternatives (a) and (b).

Case 1. $\dim\{A(L_\lambda)\} > 1$ *for some* $\lambda < \beta$.

In this case let u_1, u_2 $(\in L_\lambda)$ be chosen in such a way that Au_1, Au_2 are linearly independent, and let $e (\neq 0) \in H_\rho \ominus L_\lambda$. We now sub-divide case 1 into two alternatives.

Alternative (a). *For some $\mu > \alpha$, $(I - E_\mu)Au_1$, $(I - E_\mu)Au_2$ are linearly independent.* Let $f (\neq 0) \in L_\mu$. We may assume (by making a non-singular

transformation on u_1, u_2) that $(I-E_\mu)Au_1$, $(I-E_\mu)Au_2$ are orthogonal. We then define

$$Cx = \langle x, e\rangle u_1, \qquad Bx = \langle x, (I-E_\mu)Au_2\rangle f \qquad (x \in H_\rho).$$

It is easily verified by means of Lemma 3 that $B, C \in \mathfrak{J}_\rho$; furthermore, $BAC = 0$. However, neither BA nor AC is zero, since $ACe \neq 0 \neq BAu_2$.

Alternative (b). *For every* $\mu > \alpha$, $(I-E_\mu)Au_1$, $(I-E_\mu)Au_2$ *are linearly dependent.* We may choose scalars l_μ, m_μ $(\mu > \alpha)$ such that

$$|l_\mu| + |m_\mu| = 1, \qquad l_\mu(I-E_\mu)Au_1 + m_\mu(I-E_\mu)Au_2 = 0.$$

As $\mu \searrow \alpha$ through a suitable sequence, l_μ and m_μ tend to limits l, m (respectively), and we have

$$|l| + |m| = 1, \qquad l(I-E_\alpha)Au_1 + m(I-E_\alpha)Au_2 = 0.$$

It follows that $(I-E_\alpha)Au_1$, $(I-E_\alpha)Au_2$ are linearly dependent, and hence that $E_\alpha \neq 0$. If we now set $u_0 = lu_1 + mu_2$, then $0 \neq Au_0 = E_\alpha Au_0$. Since $E_\alpha = E_\alpha - E_{\alpha-}$, we may describe E_α as the projection of multiplication by the function taking the value unity at α and vanishing elsewhere. Thus the single point α has positive ρ-measure, and E_α is an operator of rank unity. Since $\dim\{A(L_\lambda)\} > 1$, we may choose u_3 $(\in L_\lambda)$ such that $E_\alpha Au_3 = 0 \neq Au_3$. If we now define $B = E_\alpha$, $Cx = \langle x, e\rangle u_3$ $(x \in H_\rho)$, then it is easily verified that $B, C \in \mathfrak{J}_\rho$. Furthermore, $BAC = 0$, but $ACe \neq 0 \neq BAu_0$.

Case 2. $\sup_{\lambda<\beta} \dim\{A(L_\lambda)\} = r \leqq 1$.

In this case we have $\dim\{A(L_\mu)\} = r$ for some $\mu < \beta$. Then $A(L_\lambda) = A(L_\mu)$ $(\mu \leqq \lambda < \beta)$, and hence $A(L_{\beta-}) = A(L_\mu)$ (where $L_{\beta-}$ is the range of the projection $E_{\beta-}$). We deduce that $\dim\{A(L_{\beta-})\} = r \leqq 1$, and thus that $L_{\beta-} \neq H_\rho = L_\beta$. It follows that $L_{\beta-}$ has codimension unity in H_ρ. Let $e(\neq 0) \in H_\rho \ominus L_{\beta-}$. Since H_ρ is spanned by e and $L_{\beta-}$, $A(H_\rho)$ is spanned by Ae and $A(L_{\beta-})$. From the inequalities $\dim\{A(H_\rho)\} \geqq 2$, $\dim\{A(L_{\beta-})\} \leqq 1$ we may now deduce that

$$\dim\{A(H_\rho)\} = 2, \quad \dim\{A(L_{\beta-})\} = 1, \quad Ae \notin A(L_{\beta-}).$$

Let u_0 be an element of $L_{\beta-}$ such that $Au_0 \neq 0$. We now sub-divide case 2 into two alternatives.

Alternative (a). *For some* $\mu > \alpha$, $(I-E_\mu)Au_0$, $(I-E_\mu)Ae$ *are linearly independent.* Choose $f(\neq 0) \in L_\mu$, and $u_1 \in H_\rho \ominus L_\mu$, such that

$$\langle Ae, u_1\rangle = 0 \neq \langle Au_0, u_1\rangle.$$

We then define

$$Bx = \langle x, u_1\rangle f, \qquad Cx = \langle x, e\rangle e \qquad (x \in H_\rho),$$

and may verify that $B, C \in \mathfrak{J}_\rho$, that $BAC = 0$, but that $BA \neq 0 \neq AC$.

Alternative (b). *For each* $\mu > \alpha$, $(I - E_\mu)Au_0$, $(I - E_\mu)Ae$ *are linearly dependent.* We may deduce, by the type of argument used in case 1 alternative (b), that $(I - E_\alpha)Au_0$, $(I - E_\alpha)Ae$ are linearly dependent. Hence there is a linear combination u_1 of e and u_0 such that $0 \neq Au_1 = E_\alpha Au_1$; and by following the method of case 1 alternative (b), we may construct $u_2 (\in H_\rho)$ such that $E_\alpha Au_2 = 0 \neq Au_2$. We define $B = E_\alpha$, $Cx = \langle x, e\rangle u_2$ $(x \in H_\rho)$. Once again we may verify that $B, C \in \mathfrak{J}_\rho$, that $BAC = 0$, but that $BA \neq 0 \neq AC$.

The four situations considered above are jointly exhaustive, and in each, the assumption that A has rank exceeding unity leads to a proof that condition (ii) of the lemma is not satisfied. Hence the two stated conditions are fulfilled if and only if A has rank unity. Since these conditions are obviously invariant under algebraic isomorphism, the lemma is proved.

LEMMA 5. *Let* $\mathfrak{J}$ *be an ordered basis over a separable Hilbert space* H, *and suppose that* $T \in \mathfrak{J}$. *If* $ST = 0$ *whenever* $S \in \mathfrak{J}$ *and* S *has rank unity, then* $T = 0$.

Proof. It is again sufficient to consider the case in which $\mathfrak{J} = \mathfrak{J}_\rho$, where ρ is some non-negative measure defined on the Borel subsets of $[0, 1]$. Suppose that $T \neq 0$, and choose $u (\in H_\rho)$ such that $Tu \neq 0$. We consider two cases. In the following, α is defined as in the proof of Lemma 4.

Case 1. *For some* $\lambda > \alpha$, $(I - E_\lambda)Tu \neq 0$. Choose $f (\neq 0)$ such that $f = E_\lambda f$, and define $Sx = \langle x, (I - E_\lambda)Tu\rangle f$ $(x \in H_\rho)$. Then we may verify by means of Lemma 3 that $S \in \mathfrak{J}_\rho$. Furthermore, S has rank unity and $STu \neq 0$.

Case 2. *For every* $\lambda > \alpha$, $(I - E_\lambda)Tu = 0$. Then $(I - E_\alpha)Tu = 0$, and thus $E_\alpha \neq 0$. If we now set $S = E_\alpha$, it follows that S is an operator of rank unity, and that $STu = Tu \neq 0$.

The two cases considered above are jointly exhaustive, and in each of them the assumption that $T \neq 0$ leads to the construction of an operator S of rank unity such that $S \in \mathfrak{J}_\rho$ and $ST \neq 0$. Hence the lemma is proved.

LEMMA 6. *Let* $\mathfrak{J}_1$, $\mathfrak{J}_2$ *be ordered bases over separable Hilbert spaces* H_1, H_2 *respectively. Let* ϕ, ψ *be two algebraic isomorphisms from* $\mathfrak{J}_1$ *onto* $\mathfrak{J}_2$ *such that* $\phi(S) = \psi(S)$ *whenever* $S \in \mathfrak{J}_1$ *and* S *has rank unity. Then* $\phi = \psi$.

Proof. Let $T \in \mathfrak{J}_1$, and define $T' = \phi(T) - \psi(T)$. Let S' be any operator of rank unity in $\mathfrak{J}_2$. We may set $S' = \phi(S)$, where $S \in \mathfrak{J}_1$ and (by virtue of Lemma 4) S has rank unity. Hence the operators S, ST have rank at most unity, and we have

$$\begin{aligned} \psi(S) &= \phi(S) = S', \\ 0 &= \phi(ST) - \psi(ST) \\ &= \phi(S)\phi(T) - \psi(S)\psi(T) \\ &= S'\{\phi(T) - \psi(T)\} \\ &= S'T'. \end{aligned}$$

This holds whenever $S' \in \mathfrak{J}_2$ and S' has rank unity. Lemma 5 now implies that $T' = 0$, that is, $\phi(T) = \psi(T)$. Since T is an arbitrary element of $\mathfrak{J}_1$, we have $\phi = \psi$.

4. Diagonal isomorphisms from an ordered basis onto itself. Let $\mathfrak{J}$ be an ordered basis over a separable Hilbert space H, with diagonal $\mathcal{A}$. By a *diagonal isomorphism on* $\mathfrak{J}$ we shall mean an algebraic isomorphism ϕ from $\mathfrak{J}$ onto itself, such that $\phi(A) = A$ $(A \in \mathcal{A})$.

THEOREM 1. *Let $\mathfrak{J}$ be an ordered basis over a separable Hilbert space H, with diagonal $\mathcal{A}$, and let ϕ be a diagonal isomorphism on $\mathfrak{J}$. Then ϕ is a spatial isomorphism, and may be implemented by a non-singular transformation belonging to the diagonal $\mathcal{A}$.*

Remark. For the case in which the ordered basis $\mathfrak{J}$ has an atomic diagonal, this result is contained in Theorem 3.2.3 of [3].

Proof. We may suppose without loss of generality that $\mathfrak{J}$ is the ordered basis $\mathfrak{J}_\rho$ described in § 2. We define

$$\alpha = \sup\{\lambda : \rho([0,\lambda]) = 0\}, \qquad \beta = \inf\{\lambda : \rho([\lambda,1]) = 0\}.$$

It is clear that $\rho([0,\alpha)) = 0 = \rho((\beta,1])$. We may ignore the trivial case in which the measure ρ is concentrated on a single point (and H_ρ is one-dimensional), and deduce that $\beta > \alpha$. By a linear transformation on the real numbers we may reduce to the case in which $\beta = 1$, $\alpha = 0$. Thus we have $H = H_\rho$, $\mathfrak{J} = \mathfrak{J}_\rho$, and

$$\rho([0,\lambda]) \neq 0 \neq \rho([\lambda,1]) \qquad (0 < \lambda < 1), \tag{1}$$

$$E_\lambda \neq 0 \neq I - E_\lambda. \qquad (0 < \lambda < 1). \tag{2}$$

The remainder of the proof is divided into a number of sections.

(a) *Proper dissection points.* These are the points $\lambda (\in [0,1])$ such that both the intervals $[0,\lambda]$, $[\lambda,1]$ have positive ρ-measure. Under our assumptions, every interior point of $[0,1]$ is a proper dissection point: an end-point of $[0,1]$ is a proper dissection point if and only if it has positive ρ-measure. Hence the proper dissection points form an interval X such that

$$(0,1) \subset X \subset [0,1], \qquad \rho(X) = \rho([0,1]). \tag{3}$$

(b) *The functions $a(t)$, $b(t)$.* When $e, f \in H_\rho$, we denote by $e \otimes f$ the linear operator S from H_ρ into itself which is defined by $Sx = \langle x, e\rangle f$ $(x \in H_\rho)$. When τ is a Borel subset of $[0,1]$, we write $\chi_\tau = \chi_\tau(t)$ for the characteristic function of τ.

Given a proper dissection point λ, we may consider the operator $F_\lambda = \chi_{[\lambda,1]} \otimes \chi_{[0,\lambda]}$. This operator has rank unity, and it is immediate from Lemma 3 that $F_\lambda \in \mathcal{J}_\rho$. It is easily verified that

$$F_\lambda = E_\lambda F_\lambda (I - E_{\lambda-}).$$

By applying the diagonal isomorphism ϕ, which leaves invariant the operators $I, E_\lambda, E_{\lambda-} \in \mathcal{A}_\rho$, we obtain

$$\phi(F) = E_\lambda \phi(F_\lambda)\,(I - E_{\lambda-}). \tag{4}$$

From Lemma 4 it follows that the operator $\phi(F_\lambda)$ has rank unity; we may set

$$\phi(F_\lambda) = a_\lambda \otimes b_\lambda,$$

where a_λ, b_λ are non-zero elements of H_ρ, and are determined to within scalar factors ζ, η (respectively) such that $\overline{\zeta}\eta = 1$. From (4) it follows that (possibly after modifying a_λ, b_λ on ρ-null sets) we have

$$\begin{aligned} a_\lambda(t) &= 0 \quad (t < \lambda), \\ b_\lambda(t) &= 0 \quad (t > \lambda). \end{aligned} \tag{5}$$

Our immediate objective is to show that, if a_λ and b_λ are suitably normalised, then we have

$$a_\lambda(t) = \chi_{[\lambda,1]}(t)\,a(t), \qquad b_\lambda(t) = \chi_{[0,\lambda]}(t)\,b(t) \qquad (\lambda \in X),$$

where a and b are fixed ρ-measurable functions.

Suppose therefore that $\mu, \lambda \in X$ and $\mu \geqq \lambda$. We have

$$\begin{aligned} 0 \neq{}& \chi_{[\mu,1]} \otimes \chi_{[0,\lambda]} \\ ={}& (\chi_{[\lambda,1]} \otimes \chi_{[0,\lambda]})\,(I - E_{\mu-}) \\ ={}& E_\lambda (\chi_{[\mu,1]} \otimes \chi_{[0,\mu]}). \end{aligned}$$

By applying the diagonal isomorphism ϕ we obtain

$$(a_\lambda \otimes b_\lambda)(I - E_{\mu-}) = E_\lambda(a_\mu \otimes b_\mu) \neq 0,$$

that is

$$(\chi_{[\mu,1]}a_\lambda) \otimes b_\lambda = a_\mu \otimes (\chi_{[0,\lambda]}b_\mu) \neq 0. \tag{6}$$

We now choose and fix a proper dissection point α. If $\lambda \in X$ and $\lambda \leqq \alpha$, we consider equation (6), with μ replaced by α. If $\lambda \in X$ and $\lambda \geqq \alpha$, we consider equation (6), with λ, μ replaced (respectively) by α, λ. We conclude that the functions a_λ, b_λ may be normalized in such a way that, with the possible exception of certain sets of ρ-measure zero, we have

$$\left.\begin{aligned} a_\lambda(t) &= a_\alpha(t)\chi_{[\lambda,1]}(t) \\ \chi_{[0,\alpha]}(t)b_\lambda(t) &= b_\alpha(t) \end{aligned}\right\} \quad (\lambda \in X, \lambda \geqq \alpha), \tag{7}$$

$$\left.\begin{aligned} \chi_{[\alpha,1]}(t)a_\lambda(t) &= a_\alpha(t) \\ b_\lambda(t) &= b_\alpha(t)\chi_{[0,\lambda]}(t) \end{aligned}\right\} \quad (\lambda \in X, \lambda \leqq \alpha). \tag{8}$$

Having normalised the functions a_λ, b_λ $(\lambda \in X)$ by means of equations (7), (8), we remark that

$$\left.\begin{aligned} a_\mu(t) &= a_\lambda(t)\chi_{[\mu,1]}(t) \\ \chi_{[0,\lambda]}(t)b_\mu(t) &= b_\lambda(t) \end{aligned}\right\} \quad (\lambda, \mu \in X, \lambda \leqq \mu), \tag{9}$$

with the possible exception of certain ρ-null sets. The existence of *some* normalisation of the functions a_λ, a_μ, b_λ, b_μ for which (9) is valid is an immediate consequence of (6). To prove the validity of (9) for the particular normalisation given by (7), (8), we note that both $a_\mu(t)$, $a_\lambda(t)$ are equal to $a_\alpha(t)$ at almost all points of the interval $[\max(\mu, \lambda, \alpha), 1]$: and that the three functions do not vanish at almost all points of this interval (since otherwise, in view of (5), one of them would be a ρ-null function, contrary to hypothesis). Hence the normalisation for which (9) is valid coincides with that defined by (7) and (8).

We now choose an increasing sequence of closed intervals $X_n = [\lambda_n, \mu_n]$ whose union is X, and define

$$a(t) = \begin{cases} a_{\lambda_1}(t) & (\lambda_1 \leqq t \leqq 1), \\ a_{\lambda_n}(t) & (\lambda_n \leqq t < \lambda_{n-1} : n > 1); \end{cases}$$

$$b(t) = \begin{cases} b_{\mu_1}(t) & (0 \leqq t \leqq \mu_1), \\ b_{\mu_n}(t) & (\mu_{n-1} < t \leqq \mu_n : n > 1). \end{cases}$$

The functions $a(t)$, $b(t)$ are defined throughout X, and hence at almost all points of $[0, 1]$. From (9) we deduce that, with the possible exception of certain ρ-null sets, we have

$$\left.\begin{aligned} a_\lambda(t) &= a(t)\chi_{[\lambda,1]}(t), \\ b_\lambda(t) &= b(t)\chi_{[0,\lambda]}(t) \end{aligned}\right\} \quad (\lambda \in X). \tag{10}$$

Finally, we deduce from the defining property of the functions a_λ, b_λ, and from equations (10), that

$$\phi(\chi_{[\lambda,1]} \otimes \chi_{[0,\lambda]}) = (a\chi_{[\lambda,1]} \otimes b\chi_{[0,\lambda]}) \qquad (\lambda \in X). \tag{11}$$

(c) *The relation* $b(t)\overline{a(t)} = 1$. Let $\lambda, \mu \in X$ and suppose that $\lambda \leqq \mu$. An elementary calculation shows that

$$(\chi_{[\lambda,\mu]} \otimes \chi_{[0,\lambda]})\,(\chi_{[\mu,1]} \otimes \chi_{[\lambda,\mu]}) = \rho([\lambda,\mu])\,(\chi_{[\mu,1]} \otimes \chi_{[0,\lambda]}).$$

This relation may be re-written in the form

$$(\chi_{[\lambda,1]} \otimes \chi_{[0,\lambda]}) E_{\mu-}\,(I - E_{\lambda-})\,(\chi_{[\mu,1]} \otimes \chi_{[0,\mu]}) = \rho([\lambda,\mu]) E_\lambda (\chi_{[\mu,1]} \otimes \chi_{[0,\mu]}).$$

If we apply the diagonal isomorphism ϕ to the last equation, and use (11), we obtain

$$(a\chi_{[\lambda,1]} \otimes b\chi_{[0,\lambda]}) E_{\mu-}\,(I - E_{\lambda-})\,(a\chi_{[\mu,1]} \otimes b\chi_{[0,\mu]}) = \rho([\lambda,\mu]) E_\lambda (a\chi_{[\mu,1]} \otimes b\chi_{[0,\mu]}),$$

that is

$$(a\chi_{[\lambda,\mu]} \otimes b\chi_{[0,\lambda]})\,(a\chi_{[\mu,1]} \otimes b\chi_{[\lambda,\mu]}) = \rho([\lambda,\mu])\,(a\chi_{[\mu,1]} \otimes b\chi_{[0,\lambda]}). \tag{12}$$

An elementary calculation shows that the left hand side of (12) is equal to

$$\Big(\int_{[\lambda,\mu]} b(t)\overline{a(t)}\, d\rho(t)\Big)\,(a\chi_{[\mu,1]} \otimes b\chi_{[0,\lambda]}).$$

Since, as has been verified in the above calculations,

$$(a\chi_{[\mu,1]} \otimes b\chi_{[0,\lambda]}) = \phi(\chi_{[\mu,1]} \otimes \chi_{[0,\lambda]}) \neq 0,$$

we deduce that

$$\int_{[\lambda,\mu]} b(t)\overline{a(t)}\, d\rho(t) = \rho([\lambda,\mu]) \qquad (\lambda, \mu \in X : \lambda \leqq \mu).$$

It follows that $b(t)\overline{a(t)} = 1$ at almost all points of X, and hence at almost all points of $[0,1]$.

(d) *Proof that* a, b *are* ρ*-essentially bounded.* Suppose that $f, g \in H_\rho$, that $\lambda \in X$, and that the functions $\bar{f}\chi_{[\lambda,1]}$, $g\chi_{[0,\lambda]}$ are not ρ-null functions. Then the operator $\bar{f}\chi_{[\lambda,1]} \otimes g\chi_{[0,\lambda]}$ has rank unity, and is a member of $\mathfrak{I}_\rho$. We define

$$f_1 = 1/(1+|f|), \qquad f_2 = f/(1+|f|),$$
$$g_1 = 1/(1+|g|), \qquad g_2 = g/(1+|g|).$$

Then f_1, f_2, g_1, g_2 are ρ-essentially bounded measurable functions. With these we associate the corresponding multiplication operators $M_{f_1}, \cdots, M_{g_2}$, which belong to the diagonal $\mathcal{A}_\rho$ of the ordered basis $\mathcal{J}_\rho$. From the (easily verified) formula

$$M_{h_1}(\bar{h}_2 \otimes h_3) M_{h_4} = (\bar{h}_4 h_2) \otimes (h_1 h_3) \qquad (h_1, h_4 \in L^\infty([0,1]:\rho)\,;\, h_2, h_3 \in H_\rho),$$

and the relations $ff_1 = f_2$, $gg_1 = g_2$, we deduce that

$$M_{g_1}(\bar{f}\chi_{[\lambda,1]} \otimes g\chi_{[0,\lambda]}) M_{f_1} = M_{g_2}(\chi_{[\lambda,1]} \otimes \chi_{[0,\lambda]}) M_{f_2}. \tag{13}$$

Let us now operate on equation (13) with the diagonal isomorphism ϕ. We may suppose that

$$\phi(\bar{f}\chi_{[\lambda,1]} \otimes g\chi_{[0,\lambda]}) = \bar{F} \otimes G,$$

where F, G are non-zero elements of H_ρ. We obtain

$$M_{g_1}(\bar{F} \otimes G) M_{f_1} = M_{g_2}(a\chi_{[\lambda,1]} \otimes b\chi_{[0,\lambda]}) M_{f_2},$$

that is

$$(\bar{f}_1 \bar{F}) \otimes (g_1 G) = (a\bar{f}_2 \chi_{[\lambda,1]}) \otimes (bg_2 \chi_{[0,\lambda]}).$$

It follows that F and G may be normalised in such a way that (with the possible exception of certain ρ-null sets)

$$f_1 F = \bar{a} f_2 \chi_{[\lambda,1]},$$
$$g_1 G = b g_2 \chi_{[0,\lambda]}.$$

Hence

$$F = \bar{a} f \chi_{[\lambda,1]},$$
$$G = bg\chi_{[0,\lambda]}.$$

We deduce that $\bar{a} f \chi_{[\lambda,1]}$, $bg\chi_{[0,\lambda]} \in H_\rho$, and that

$$\phi(\bar{f}\chi_{[\lambda,1]} \otimes g\chi_{[0,\lambda]}) = (a\bar{f}\chi_{[\lambda,1]}) \otimes (bg\chi_{[0,\lambda]}). \tag{14}$$

It is apparent that this equation remains valid when one or both of $\bar{f}\chi_{[\lambda,1]}$, $g\chi_{[0,\lambda]}$ is a ρ-null function.

We now choose a proper dissection point λ, and an element f of H_ρ, such that $f(t) \neq 0$ $(0 \leqq t \leqq 1)$. We keep f and λ fixed, but allow g to vary in H_ρ. Since ϕ is an algebraic isomorphism from $\mathcal{J}_\rho$ onto itself, and preserves the property of being of rank unity, we deduce from equation (14) that as g runs through H_ρ, $bg\chi_{[0,\lambda]}$ runs through all elements of H_ρ which vanish on the interval $(\lambda, 1]$. It follows that both b and its inverse function b^{-1} are ρ-essentially bounded on $[0, \lambda]$. Similar considerations show that a and a^{-1} are ρ-essentially bounded on $[\lambda, 1]$. Since $b\bar{a} = 1$, we may deduce that a, a^{-1} are ρ-essentially bounded on $[0, 1]$.

(e) *Conclusion of the proof of Theorem* 1. Equation (14) may be written in the form

$$\phi(\bar{f}\chi_{[\lambda,1]} \otimes g\chi_{[0,\lambda]}) = M_b(\bar{f}\chi_{[\lambda,1]} \otimes g\chi_{[0,\lambda]})M_{\tilde{a}} = Q^{-1}(\bar{f}\chi_{[\lambda,1]} \otimes g\chi_{[0,\lambda]})Q, \tag{15}$$

where Q ($\in \mathcal{A}_\rho$) is the operator $M_{\tilde{a}}$. It follows from Lemma 3 that every operator of rank unity in $\mathcal{J}_\rho$ can be written in the form $\bar{f}\chi_{[\lambda,1]} \otimes g\chi_{[0,\lambda]}$ for some $\lambda \in X$. From (15) we deduce that ϕ coincides, in its action on operators of rank unity, with the spatial isomorphism induced by Q. It now follows from Lemma 6 that ϕ coincides with this spatial isomorphism. This completes the proof of Thorem 1.

5. Symmetric isomorphisms between ordered bases. Let $\mathcal{J}_i$ be an ordered basis over a separable Hilbert space H_i, with diagonal $\mathcal{A}_i$ ($i = 1, 2$). By a *symmetric isomorphism* from $\mathcal{J}_1$ onto $\mathcal{J}_2$ we shall mean an algebraic isomorphism ϕ from $\mathcal{J}_1$ onto $\mathcal{J}_2$ such that $\phi(A^*) = \phi(A)^*$ ($A \in \mathcal{A}_1$).

THEOREM 2. *Let $\mathcal{J}_i$ be an ordered basis over a separable Hilbert space H_i, with diagonal $\mathcal{A}_i$ ($i = 1, 2$), and let ϕ be a symmetric isomorphism from $\mathcal{J}_1$ onto $\mathcal{J}_2$. Then ϕ is a spatial isomorphism, and may be implemented by a transformation of the form $Q = Q_1U$, where U is a unitary operator from H_2 onto H_1 and $Q_1 \in \mathcal{A}_1$.*

Remark. For the case in which each of the ordered bases has atomic diagonal, this result is contained in Theorem 3.2.3 of [3]. I am indebted to Professor R. V. Kadison for a suggestion which has enabled me to shorten substantially the arguments by which Theorem 2 is deduced from Theorem 1.

Proof. Let $A \in \mathcal{A}_1$. Then $A, A^* \in \mathcal{J}_1$ and we have

$$\phi(A) \in \mathcal{J}_2, \qquad \phi(A)^* = \phi(A^*) \in \mathcal{J}_2.$$

It follows that $\phi(A) \in \mathcal{A}_2$, and that $\phi(\mathcal{A}_1) \subset \mathcal{A}_2$. Since $\phi(\mathcal{A}_1)$ is a maximal abelian sub-algebra of $\mathcal{J}_2$, while $\mathcal{A}_2$ is abelian, we may deduce that $\phi(\mathcal{A}_1) = \mathcal{A}_2$. Thus ϕ induces an adjoint-preserving algebraic isomorphism from $\mathcal{A}_1$ onto $\mathcal{A}_2$; and $\mathcal{A}_i$ is a maximal abelian self-adjoint sub-algebra of (H_i, H_i) ($i = 1, 2$). It follows (see the uniqueness clause of Theorem 5 of [5]) that there is a unitary operator U from H_2 onto H_1 such that

$$\phi(A) = U^{-1}AU \qquad (A \in \mathcal{A}_1). \tag{16}$$

We shall now prove that

$$\mathcal{J}_2 = U^{-1}\mathcal{J}_1U. \tag{17}$$

For let $\mathfrak{F}_1$ be the family of all hull projections in $\mathfrak{J}_1$. Then $\mathfrak{F}_1 \subset \mathfrak{A}_1$, and $\mathfrak{F}_1$ is a full set of hull idempotents for $\mathfrak{J}_1$. If we now define

$$\mathfrak{F}_2 = \phi(\mathfrak{F}_1) = U^{-1}\mathfrak{F}_1 U, \tag{18}$$

it follows from Lemma 2 that $\mathfrak{F}_2$ is a full set of hull idempotents for $\mathfrak{J}_2$. Hence, for $i = 1, 2$, $\mathfrak{J}_i$ consists of all operators S which act in H_i and satisfy $SE = ESE$ $(E \in \mathfrak{F}_i)$. From (18) it is immediate that $\mathfrak{J}_2 = U^{-1}\mathfrak{J}_1 U$. This proves (17).

We now define

$$\psi(S) = U\phi(S)U^{-1} \qquad (S \in \mathfrak{J}_1). \tag{19}$$

From (16) and (17) we deduce that ψ is an algebraic isomorphism from $\mathfrak{J}_1$ onto itself, and that $\psi(A) = A$ $(A \in \mathfrak{A}_1)$. Thus ψ is a diagonal isomorphism from $\mathfrak{J}_1$ onto itself, and (by Thesorem 1) is a spatial isomorphism which may be implemented by an operator Q_1 $(\in \mathfrak{A}_1)$. Hence

$$\psi(S) = Q_1^{-1} S Q_1 \qquad (S \in \mathfrak{J}_1),$$

and

$$\begin{aligned} \phi(S) &= U^{-1}\psi(S)U \\ &= U^{-1}Q_1^{-1}SQ_1U \\ &= Q^{-1}SQ \qquad (S \in \mathfrak{J}_1), \end{aligned}$$

where $Q = Q_1 U$. This proves Theorem 2.

6. Algebraic isomorphisms between ordered bases.

THEOREM 3. *Let $\mathfrak{J}_i$ be an ordered basis over a separable Hilbert space H_i $(i = 1, 2)$, and let ϕ be an algebraic isomorphism from $\mathfrak{J}_1$ onto $\mathfrak{J}_2$. Then ϕ is a spatial isomorphism.*

Proof. In view of the representation theorem stated in § 2, it is sufficient to consider the case in which $\mathfrak{J}_1 = \mathfrak{J}_\rho$, where ρ is some non-negative measure defined on the Borel subsets of $[0, 1]$. We shall use the notations introduced in § 2. The proof is divided into a number of stages.

(a) Consider the algebra $\phi(\mathfrak{A}_\rho)$. This is a maximal abelian sub-algebra of $\mathfrak{J}_2$, and since $\mathfrak{J}_2$ is weakly closed, the same is true of $\phi(\mathfrak{A}_\rho)$. A fortiori $\phi(\mathfrak{A}_\rho)$ is uniformly closed and, with operator bound as norm, forms a commutative Banach algebra. Thus ϕ induces an algebraic isomorphism ψ from the commutative von Neumann algebra $\mathfrak{A}_\rho$ onto the commutative Banach algebra $\phi(\mathfrak{A}_\rho)$. It is clear, from the definition

$$r_A = \inf\{\mu : \mu > 0, |\lambda| > \mu \Rightarrow \lambda I - A \text{ has an inverse in } \mathfrak{J}\}$$

of the *spectral radius* r_A of an element A in a Banach algebra $\mathcal{T}$, that ψ preserves spectral radius. Now

$$r_A = \| A \| \qquad (A \in \mathcal{A}_\rho),$$

since $\mathcal{A}_\rho$ consists of normal operators; and

$$r_{\psi(A)} \leqq \| \psi(A) \| \qquad (A \in \mathcal{A}_\rho),$$

by a well known inequality (see, for example, [2], p. 567, Lemma 4). Thus we have

$$\| A \| = r_A = r_{\psi(A)} \leqq \| \psi(A) \| \qquad (A \in \mathcal{A}_\rho).$$

Hence (considering $\mathcal{A}_\rho$ and $\phi(\mathcal{A}_\rho)$ as Banach spaces) ψ^{-1} is a continuous linear operator from $\phi(\mathcal{A}_\rho)$ onto $\mathcal{A}_\rho$. The inversion theorem (see, for example, [2], p. 57, Theorem 2) implies that ψ is also continuous. We may therefore choose a constant K such that

$$\| A \| \leqq \| \phi(A) \| \leqq K \| A \| \qquad (A \in \mathcal{A}_\rho). \tag{20}$$

(b) Let $E(\tau)$ be the spectral measure in $\mathcal{A}_\rho$, and let E_λ, $E_{\lambda-}$ be defined as in § 2. We set

$$\begin{aligned} F(\tau) &= \phi(E(\tau)), \\ F_\lambda &= \phi(E_\lambda), \\ F_{\lambda-} &= \phi(E_{\lambda-}) \end{aligned} \tag{21}$$

for all Borel subsets τ of $[0,1]$ and all $\lambda \in [0,1]$. It is immediate that

$$F_\lambda = F([0,\lambda]), \quad F_{\lambda-} = F([0,\lambda)) \qquad (0 \leqq \lambda \leqq 1). \tag{22}$$

Furthermore, if τ, τ_1, τ_2 are Borel subsets of $[0,1]$, τ_0 is the empty set, and K is the constant occuring in (20), we have

(i) $F(\tau_0) = 0, \quad F([0,1]) = I;$

(ii) $F(\tau_1 \cap \tau_2) = F(\tau_1)F(\tau_2),$
$F(\tau_1 \cup \tau_2) = F(\tau_1) + F(\tau_2)$ provided $\tau_1 \cap \tau_2 = \tau_0;$

(iii) $\| F(\tau) \| \leqq K.$

Conditions (i), (ii) follow from the corresponding properties of $E(\tau)$, and (iii) is seen to be a consequence of (20) and (21) when we recall that $E(\tau) \in \mathcal{A}_\rho$ and that $\| E(\tau) \| \leqq 1$.

We may now deduce from a theorem of Dixmier ([1], p. 222: see also [4], [6]) that it is possible to find a Hilbert space H_3, and a one to one bicontinuous linear operator P from H_3 onto H_2, such that

$$G(\tau) = P^{-1}F(\tau)P \tag{23}$$

is a self-adjoint operator for each Borel subset τ of $[0,1]$. It is clear that H_3 is separable.

(c) We define

$$\mathcal{J}_3 = P^{-1}\mathcal{J}_2 P, \qquad \mathcal{A}_3 = P^{-1}\phi(\mathcal{A}_\rho)P.$$

We shall prove that $\mathcal{J}_3$ is an ordered basis with diagonal $\mathcal{A}_3$.

Since the operators E_λ, $E_{\lambda-}$ $(0 \leqq \lambda \leqq 1)$ form a full set of hull idempotents for $\mathcal{J}_\rho$, it follows from Lemma 2 and from equations (21) that F_λ, $F_{\lambda-}$ $(0 \leqq \lambda \leqq 1)$ are a full set of hull idempotents for $\mathcal{J}_2$. Hence $\mathcal{J}_2$ consists of all operators acting in H_2 which leave invariant the ranges of the idempotents F_λ, $F_{\lambda-}$ $(0 \leqq \lambda \leqq 1)$. Thus, if we define

$$G_\lambda = P^{-1}F_\lambda P, \qquad G_{\lambda-} = P^{-1}F_{\lambda-}P, \tag{24}$$

then $\mathcal{J}_3$ consists of all operators acting in H_3 which leave invariant the ranges of the idempotents G_λ, $G_{\lambda-}$ $(0 \leqq \lambda \leqq 1)$. Hence

$$\mathcal{J}_3 = \{T \in (H_3, H_3) : TG_\lambda = G_\lambda TG_\lambda,\ TG_{\lambda-} = G_{\lambda-}TG_{\lambda-} \ (0 \leqq \lambda \leqq 1)\}. \tag{25}$$

From equations (22), (23) and (24) we deduce that $G_\lambda = G([0,\lambda])$, $G_{\lambda-} = G([0,\lambda))$. Hence the idempotents G_λ, $G_{\lambda-}$ $(0 \leqq \lambda \leqq 1)$ are projections, and in fact form a linearly ordered family of projections. If we now prove that the von Neumann algebra $\mathcal{B}_3$ generated by these projections is a maximal abelian sub-algebra of (H_3, H_3), we shall be able to deduce from Theorem 3.1.1 of [3] that $\mathcal{J}_3$ is an ordered basis with diagonal $\mathcal{B}_3$.

When $S \in \mathcal{J}_\rho$ we define

$$\psi(S) = P^{-1}\phi(S)P. \tag{26}$$

Thus ψ is an algebraic isomorphism from $\mathcal{J}_\rho$ onto $\mathcal{J}_3$, and

$$\psi(\mathcal{A}_\rho) = P^{-1}\phi(\mathcal{A}_\rho)P = \mathcal{A}_3. \tag{27}$$

Furthermore,

$$\psi(E_\lambda) = G_\lambda, \qquad \psi(E_{\lambda-}) = G_{\lambda-} \qquad (0 \leqq \lambda \leqq 1). \tag{28}$$

Let $\mathcal{B}'_3$ denote the commutant of $\mathcal{B}_3$. Then

$$\mathcal{B}'_3 = \{T \in (H_3, H_3) : TG_\lambda = G_\lambda T, \quad TG_{\lambda-} = G_{\lambda-}T \qquad (0 \leqq \lambda \leqq 1)\}.$$

It is clear from (25), and the preceding line, that $\mathcal{B}'_3 \subset \mathcal{J}_3$; hence

$$\begin{aligned}\mathcal{B}'_3 &= \{T \in \mathcal{J}_3 : TG_\lambda = G_\lambda T, TG_{\lambda-} = G_{\lambda-}T \ (0 \leqq \lambda \leqq 1)\}\\ &= \psi(\{S \in \mathcal{J}_\rho : SE_\lambda = E_\lambda S, SE_{\lambda-} = E_{\lambda-}S \ (0 \leqq \lambda \leqq 1)\})\\ &= \psi(\mathcal{A}_\rho)\\ &= \mathcal{A}_3.\end{aligned}$$

Thus $\mathcal{B}'_3$ is abelian, and contains $\mathcal{B}_3$ (since $G_\lambda, G_{\lambda-} \in \mathcal{A}_3$). Since, by the double commutant theorem, $\mathcal{B}_3 = \mathcal{B}''_3$, we deduce that $\mathcal{B}_3 = (\mathcal{B}'_3)' \subset \mathcal{B}'_3$. Hence $\mathcal{B}_3 = \mathcal{B}'_3 = \mathcal{A}_3$. It easily follows that $\mathcal{B}_3$ $(= \mathcal{A}_3)$ is a maximal abelian sub-algebra of (H_3, H_3), and hence that $\mathcal{T}_3$ is an ordered basis with diagonal $\mathcal{A}_3$.

(d) From (20) and (26) it follows that ψ, as a mapping from $\mathcal{A}_\rho$ onto $\mathcal{A}_3$, is continuous if each of these algebras is considered with its uniform topology. Since

$$\psi(E(\tau)) = G(\tau),$$

we deduce that ψ is adjoint-preserving on the uniformly closed sub-algebra of $\mathcal{A}_\rho$ which is generated by the projections $E(\tau)$. But this sub-algebra is in fact the whole of $\mathcal{A}_\rho$ (as follows from the spectral theorem for normal operators, or by elementary arguments based on the fact that $\mathcal{A}_\rho$ is a multiplication algebra). Hence

$$\psi(T^*) = \psi(T)^* \qquad (T \in \mathcal{A}_\rho).$$

Thus ψ is a symmetric isomorphism from $\mathcal{T}_\rho$ onto $\mathcal{T}_3$, and, by Theorem 2, is spatial. From equation (26) it is now immediate that ϕ is spatial. This proves Theorem 3.

UNIVERSITY OF DURHAM,
KING'S COLLEGE,
NEWCASTLE UPON TYNE 2, ENGLAND.

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SOME GLOBAL PROBLEMS IN THE THEORY OF FUNCTIONS OF SEVERAL COMPLEX VARIABLES.*

By ERRETT BISHOP.[1]

1. Introduction. In a previous paper [2] the global structure of a complex analytic variety was examined in some detail. The present paper is a sequel containing some application of those results. In order that this paper be self-contained, the definitions and results of [2] are summarized in Section 2.

Consider now an analytic subset L of a Stein manifold K. It is a consequence of Theorems A and B of [3] that

I. The set L is the set of common zeros of some family Γ of analytic functions on K.

II. If L is a submanifold of K then every analytic function on L can be extended to an analytic function on K.

Previous work on Problems I and II was done by Oka [5], [6], and others. In Section 3 we give new proofs of I and II which are independent of sheaf theory. The proof of I contains an explicit formula for the functions in the class Γ, and in this respect gives information which is not available from the sheaf-theoretic proof. Similarly the proof of II contains a specific formula for the extension of a function analytic on L to a function analytic on K. This formula makes it clear that II continues to hold for analytic functions whose values lie in a Banach space, thus answering a question posed by H. Rossi in a conversation with the author.

In Section 4 we prove that the first cohomology group of the sheaf of germs of holomorphic functions on a Stein manifold is trivial. Again this result follows from sheaf theory [3]. Previous work was done by Cousin [4], Oka [5], [6], [7], and others. Since it is the purpose of this paper to proceed independently of sheaf theory, this result is formulated and proved without mention of sheaves or cohomology. As is well-known, the solutions to Cousin's first and second problems are consquences of this result.

The techniques which we use for solving these problems are all essentially similar. Since all of the problems considered have previously been solved,

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we have not attempted to push the theory to the limit of its generality. It should however be clear to someone familiar with the subject that the methods employed here are capable of handling the same problems in greater generality. These methods, which seem not to afford so powerful a general approach to these questions as does the theory of sheaves, nevertheless have the advantages of explicitness and simplicity. The gain in explicitness comes from the use of explicit formulas to construct the various analytic functions which arise. These formulas make it clear, for instance, that all of the above mentioned results hold for Banach space valued analytic functions.

2. Preliminaries. In order not to burden the exposition with too much generality (as was done in [2]), we confine our attention to Stein manifolds, merely remarking that there are obvious possibilities for generalizing the results to one type or another of analytic space.

Definition 1. A *Stein manifold* K of dimension n is an n-dimensional separable complex analytic manifold K having the following properties, where $A(K)$ denotes the set of analytic functions on K

(a) For each pair (p, q) of distinct points in K there exists f in $A(K)$ with $f(p) \neq f(q)$.

(b) For each p in K there exist $f_1, \cdots, f_n$ in $A(K)$ which map some neighborhood of p homeomorphically onto an open set in n-dimensional complex affine space C^n.

(c) K is holomorphically convex. This means that for each compact subset S of K the set

$$\hat{S} = \{p: p \in K, |f(p)| \leqq \sup\{|f(q)|: q \in S\} \text{ for all } f \text{ in } A(K)\}$$

is compact.

We now state, for the special case of a Stein manifold, some of the definitions and results of [2].

Definition 2. Let K be a Stein manifold and L an analytic subset of K. Let $f_1, \cdots, f_k$ be functions in $A(K)$. Write

$$P_0 = \{p \in L: |f_i(p)| < 1, 1 \leqq i \leqq k\}.$$

Let P be an open and closed subset of P_0 such that P is a compact subset of L. Then P is called an analytic polyhedron in L defined by $f_1, \cdots, f_k$. The analytic polyhedron P is called *special* if L has the dimension k at each of its points.

THEOREM 1. *If P is an analytic polyhedron in a homogeneous-dimensional analytic subset L of a Stein manifold K, if S is a compact subset of P, and if T is a compact subset of K at each point of which each of the defining functions for P has absolute value less than* 1, *then there exists a special analytic polyhedron Q in L with*

$$S \subset Q \subset P$$

such that each of the functions defining Q has absolute value less than 1 *at each point of T. The functions defining Q can be chosen to be polynomials in the functions defining P.*

Since it is easy that a homogeneous-dimensional analytic subset L of a Stein manifold K can be exhausted by an increasing sequence of analytic polyhedra, it follows from Theorem 1 that L can be exhausted by an increasing sequence of special analytic polyhedra. Thus the study of L can be reduced for most purposes to the study of special analytic polyhedra in L. It turns out that a special analytic ploy hedron P can be studied in detail by means of the map f of P into d-dimensional complex affine space C^d, where $d = \dim L$ and f is defined by

$$f(p) = (f_1(p), \cdots, f_d(p))$$

with $f_1, \cdots, f_d$ the functions which occur in the definition of the special analytic polyhedron P. Before we state the main results about special analytic polyhedra another definition is necessary.

Definition 3. Let X be a topological space. Let ${}^kX$ be the k-fold Cartesian product of X with itself. Define an equivalence relation on ${}^kX$ by taking $(x_1, \cdots, x_k) \sim (y_1, \cdots, y_k)$ to mean that $(y_1, \cdots, y_k)$ is a reordering of $(x_1, \cdots, x_k)$. Then ${}_kX$, the k-fold *unordered product* of X with itself, is the set of equivalence classes of ${}^kX$ under this relation. A set in ${}_kX$ is called *open* if and only if its inverse image in ${}^kX$ with respect to the natural map $\pi: {}^kX \to {}_kX$ is open. The image under π of the element $(x_1, \cdots, x_k)$ in ${}^kX$ will be denoted by $\{x_1, \cdots, x_k\}$.

The following three theorems give the fundamental facts about analytic polyhedra.

THEOREM 2. *Let P be a special analytic polyhedron in a d-dimensional subset L of a Stein manifold K, defined by functions $f_1, \cdots, f_d$ in $A(K)$. Let f be the map*

$$p \to (f_1(p), \cdots, f_d(p))$$

of K into C^d, so that $f(P) \subset E^d$, where

$$E^d = \{(z_1, \cdots, z_d) : |z_i| < 1, 1 \leqq i \leqq d\}.$$

Then there exists an integer λ, a continuous map ω from E^d into ${}_\lambda P$, and a closed nowhere dense subset H of E^d such that

(a) For all z in E^d the points $p_1, \cdots, p_\lambda$ in $\omega(z) = \{p_1, \cdots, p_\lambda\}$ are mapped by f into z.

(b) No other points in P are mapped by f into z.

(c) For all z in $E^d - H$ the points $p_1, \cdots, p_\lambda$ are distinct.

THEOREM 3. *With the conditions of Theorem 2 let J be the subset of P consisting of all regular points p of P such that the functions $f_1, \cdots, f_d$ are a coordinate set of functions at each of the points $p_1, \cdots, p_\lambda$ of $\omega(f(p)) = \{p_1, \cdots, p_\lambda\}$. Let g be a continuous function on P which is analytic at each point of J. Then g is analytic at each regular point of P. If also $g(p_1) = g(p_2)$ whenever p_1 and p_2 are points in P for which $f(p_1) = f(p_2)$ and if g_0 is a function on an open subset U of $f^{-1}(E^d)$ such that $g_0(q) = g(p)$ whenever $q \in U$, $p \in P$, and $f(p) = f(q)$, then g_0 is analytic on U and the function h on E^d for which $g = h \circ f$ is analytic on E^d.*

THEOREM 4. *Let T be a compact subset of a Stein manifold K and U a neighborhood of T such that for each p in $U - T$ there exists f in $A(K)$ with*

$$|f(p)| > \sup\{|f(q)| : q \in T\}.$$

Let $C(T)$ be the uniformly-normed Banach space of all continuous complex-valued functions on T. Let $\mathfrak{A}$ be the closure of $A(K)$ in the space $C(T)$. Then the spectrum of the Banach algebra $\mathfrak{A}$ is T. That is, to any continuous homomorphism ϕ of $\mathfrak{A}$ into the complex numbers corresponds a point p_0 in T with $\phi(f) = f(p_0)$ for all f in $\mathfrak{A}$.

The proof of Theorem 4 (Lemma 6 of [2]) was for certain special sets T, but by means of Theorem 1 one can easily approximate T by such special sets and thereby obtain Theorem 4 for T.

In order to study a d-dimensional analytic subset L of a Stein manifold K globally it is sometimes necessary to consider a map of K into C^d whose restriction to L is *almost proper*. This concept is explained in the following definition.

Definition 4. A continuous map f of a topological space X into a topological space Y is called *almost proper* if every component of $f^{-1}(S)$ is compact for every compact subset S of Y.

The following theorem is easily proved from the fact that an analytic subset of a Stein manifold can be exhausted by special analytic polyhedra.

THEOREM 5. *If L is a d-dimensional analytic subset of a Stein manifold K then K admits an analytic map into C^d whose restriction to L is almost proper. The mapping can be chosen to be non-singular on L at each point of any 0-dimensional set of regular points of L.*

The final result we shall need guarantees the existence of a function f_{d+1} which will be used as a convergence factor for constructions involving an almost proper map f of an analytic subset L of a Stein manifold.

THEOREM 6. *Let $f=(f_1,\cdots,f_d)$ be an analytic map of a Stein manifold K into C^d, whose restriction to the analytic subset L of K is almost proper. Let L have dimension d at each point. Let $\{S_k\}$ be an increasing sequence of compact subsets of L, each of which has finitely many components, the union of whose interiors is L. Let α be a function from the positive integers to the positive real numbers such that*

$$S_k \subset F_k = \{p \in L : |f_i(p)| \leqq \alpha(k), 1 \leqq i \leqq d\}$$

for all k. Let H_k be the union of all those components of F_k which intersect S_k. Then H_k is compact. For each positive integer k write

$$G_k = (H_{k+1} \cap F_k) - H_k.$$

Let $G_0 = H_1$. Let $\{g_k\}$ be a sequence of functions in $A(K)$ and $\{\epsilon_k\}$ a sequence of positive real numbers. Then there exists f_{d+1} in $A(K)$ such that

$$|f_{d+1}(p) - g_k(p)| < \epsilon_k \qquad p \in G_k, \qquad 0 \leqq k < \infty.$$

In addition f_{d+1} can be chosen to have distinct values modulo $2\pi i$ at the points of any countable subset of K.

Since it is not clear that Theorem 6 as stated follows from the results of [2], we indicate the modifications necessary in the proof of Theorem 8 of [2] to give Theorem 6. The only point in question is the convergence on K (and not just on L) of the series

$$f_{d+1} = \sum_{k=0}^{\infty} g_k$$

used to define f_{d+1} on p. 238 of [2]. To see that this series can be made to converge on K, let D be any compact subset of K, with $D = \tilde{D}$. We shall show that for k sufficiently large the g_k can be chosen so that $|g_k(p)| < 2^{-k}$ for all p in D. It will be clear that this can actually be achieved for any sequence of such compact sets D which exhausts K, from which the uniform

convergence of the series for f_{d+1} on all compact subsets of K can be obtained. To this end it is sufficiently to show that the function τ constructed in the proof of Theorem 8 of [2] and used in the construction of g_k can be made arbitrarily small on D if k is sufficiently large. That is, we wish to find τ in $\mathfrak{A}$ which is arbitrarily close to 0 on $H_k \cup D$ and arbitrarily close to 1 on G_k.

Choose k so large that $D \cap L \subset H_k$. Thus there exists a neighborhood U in K of $T = H_k \cup G_k \cup D$ such that $H_k \cup G_k = F_k \cap U$. Clearly G_k is open in the set T. Thus to show that τ exists with the desired properties it is sufficient by Theorem 4 above to show that for each p in $U - T$ there exists h in $\mathfrak{A}$ with

$$|h(p)| > \sup\{|h(q)| : q \in T\}.$$

If $p \in U \cap L$ then $p \in L - F_k$ so that h can be taken to be one of the functions $f_1, \cdots, f_d$. Thus we need only consider the case $p \in U - L$. It is well known, and a new proof will be given below (Theorem 7), that there exists an analytic function h_1 on an arbitrary open relatively compact set $V \subset K$ such that $h_1(p) = 1$ and h_1 vanishes on $V \cap L$. Now since $D = \tilde{D}$ there exists h_2 in $\mathfrak{A}$ with $h_2(p) = 1$ and $|h_1(q)h_2(q)| < 1$ for all q in D. Now if V is sufficiently large then by Runge's theorem for several complex variables (a new proof of which will be given below which does not depend on this theorem), the function h_1h_2 can be uniformly approximated on $T \cup \{p\}$ by functions h in $\mathfrak{A}$, so that h will have the desired property. This completes the proof that the desired function τ can be found.

3. Analytic sets in Stein manifolds. We wish to prove Theorems I and II of the introduction. We first prove somewhat weaker results, I_1 and II_1 below, which are sufficient for most applications.

THEOREM 7. *Let L be an analytic subset of a Stein manifold K. Let U be an open subset of K whose closure is compact. Let L have the same dimension d at each of its points. Then*

I_1. *For each p_0 in $U - L$ there exists an analytic function h on U, defined by (*) below, which vanishes on $U \cap L$, with $h(p_0) \neq 0$.*

II_1. *If in addition L is a submanifold of K then every analytic function u on L can be extended to an analytic function $\tilde{u}$ on U, given by* (a), (d), *and* (e) *below.*

Proof. Since K is a Stein manifold and $\bar{U}$ is compact there exists an analytic polyhedron Q in K with $\bar{U} \subset Q$. Thus $Q \cap L$ is an analytic polyhedron in L. By Theorem 1 there exist functions $f_1, \cdots, f_d$ in $A(K)$ defining

an analytic polyhedron P in L with $U \cap L \subset P$ and $|f_i(p)| < 1$ for p in $\bar{U}$ and $1 \leqq i \leqq d$. Let f be the map

$$p \to (f_1(p), \cdots, f_d(p))$$

of U into E^d and let f_0 be the restriction of f to P. Let λ and ω respectively be the multiplicity and multiplicity function, as defined in Theorem 2, of the map f_0 of P into E^d. Consider p_0 in $U - L$ and write $z_0 = f(p_0)$. Since K is a Stein manifold there exists w in $A(K)$ such that $w(p_0) \neq w(p)$ for all p in $\omega(z_0)$.

For an arbitrary point q in U let $\omega(f(q)) = \{q_1, \cdots, q_\lambda\}$. Define the function h on U by

$$(*) \qquad h(q) = \prod_{i=1}^{\lambda} (w(q) - w(q_i)).$$

Thus h is a polynomial in w whose coefficients a_j, $0 \leqq j \leqq \lambda$, are symmetric functions of the q_i. Clearly a_j is a continuous function on P which is analytic on L at each point of the set J defined in Theorem 3. By Theorem 3 the functions a_j are analytic on U. It follows that h is analytic on U. Clearly $h(p_0) \neq 0$ and h vanishes on $L \cap U \subset P$. This proves I_1.

To prove II_1, define $f_1, \cdots, f_k$ and P as above. Choose r_0 with $0 < r_0 < 1$ such that $|f_i(p)| < r_0$, $1 \leqq i \leqq d$, $p \in \bar{U}$. Choose a real number r with $r_0 < r < 1$. Write

$$T = \{p \in P : |f_i(p)| \leqq r, 1 \leqq i \leqq d\}$$

whenever $0 < r < 1$. Let p_0 be any point of T. Since K is a Stein manifold there exist $h_1, \cdots, h_d$ in $A(K)$ whose restrictions to L are a coordinate set of functions on L at the point p_0. Thus for all sufficiently small $\epsilon \neq 0$ the functions

$$g_i = f_i + \epsilon h_i \qquad 1 \leqq i \leqq d$$

are a coordinate set of functions on L at the point p_0. It follows from the choice of r_0 that $|g_i(p)| < r_0$, $1 \leqq i \leqq d$, for all p in $\bar{U}$ if ϵ is small enough. Now fix r_1 with $r < r_1 < 1$. Consider the sets

$$R_0 = \{p \in P : |g_i(p)| < r_0, 1 \leqq i \leqq d\}$$

and

$$R_1 = \{p \in P : |g_i(p)| < r_1, 1 \leqq i \leqq d\}.$$

If ϵ is small enough then R_0 and R_1 are analytic polyhedra in L defined by multiples of the functions $g_1, \cdots, g_d$ with $R_0 \subset T \subset R_1 \subset P$. Let g be the map $(g_1, \cdots, g_d)$ of K into C^d and let $\tilde{g}$ be the restriction of g to R_1. From Theorem 2 it follows that the map $\tilde{g}$ of R_1 into $r_1 E^d$ has a definite multiplicity

λ and multiplicity function ω. Since $g_1, \cdots, g_d$ form a coordinate system at p_0, the point p_0 occurs with multiplicity 1 in the set $\omega(g(p_0))$. Let w be any function in $A(K)$ whose value at p_0 is different from its values at the other points of $\omega(g(p_0))$.

For an arbitrary q in $g^{-1}(r_1 E^d)$ let $q_1, \cdots, q_\lambda$ be the points of the set $\omega(g(q)) = \{q_1, \cdots, q_\lambda\}$, counted according to multiplicity. Let u be any function analytic on R_0. Define the function $\tilde{u}$ on $g^{-1}(r_0 E^d) \supset U$ by

$$\text{(a)} \qquad \tilde{u}(q) = \sum_{i=1}^{\lambda} [u(q_i) \prod_{j \neq i} (w(q) - w(q_j))].$$

Now $\tilde{u}(q)$ is a polynomial in $w(q)$ whose coefficients $a_0(q), \cdots, a_\lambda(q)$ are analytic functions of q at each point q in $g^{-1}(r_0 E^d)$ for which $q_1, \cdots, q_\lambda$ are disinct points of R_0. By Theorem 3, the functions a_i are analytic on U. Therefore $\tilde{u}$ is analytic on U. For q in $U \cap L \subset R_0$ the point q is one of the points $q_1, \cdots, q_\lambda$ in $\omega(g(q))$, say $q = q_1$. In this case it is clear that

$$\text{(b)} \qquad \tilde{u}(q) = u(q)\tilde{w}(q)$$

where

$$\text{(c)} \qquad \tilde{w}(q) = \prod_{i=2}^{\lambda} (w(q) - w(q_i))$$

is an analytic function on R_1, it being understood that q_1 is chosen to equal q for q in R_1. Remembering that p_0 occurs with multiplicity 1 in the set $\omega(g(p_0))$ and that w takes on values other than $w(p_0)$ at the other points of $\omega(g(p_0))$, we see that $\tilde{w}(p_0) \neq 0$.

Thus for each p_0 in T there exists a function $\tilde{w}$ analytic on R_1 with $\tilde{w}(p_0) \neq 0$ such that for every analytic function u on R_0 there exists an analytic function $\tilde{u}$ on U which agrees with $u\tilde{w}$ on $U \cap L$. Let B be the uniform closure on T of all functions analytic on R_1. From Theorem 4 we see that the spectrum of B is T. Since the function $\tilde{w}$, which is analytic on R_1, can be chosen to be non-zero at any given point of T, there exist a finite number $\tilde{w}_1, \cdots, \tilde{w}_N$ of such functions which have no common zero in T. Thus from the general theory of Banach algebras there exist elements $c_1, \cdots, c_N$ of B with

$$\text{(d)} \qquad c_1\tilde{w}_1 + \cdots + c_N\tilde{w}_N = 1.$$

For each j, $1 \leqq j \leqq N$, let $\tilde{u}_j$ be the function defined by (a) with w replaced by w_j and u replaced by $c_j u$. Thus

$$\text{(e)} \qquad \tilde{u} = \sum_{j=1}^{N} \tilde{u}_j$$

is analytic on U, and for all q in $U \cap L$ we have, by (b), (d), and (e)

$$\begin{aligned} \tilde{u}(q) &= \sum \tilde{u}_j(q) = \sum c_j(q) u(q) \tilde{w}_j(q) \\ &= [\sum c_j(q) \tilde{w}_j(q)] u(q) = u(q). \end{aligned}$$

This completes the proof of II_1.

Remark. Theorem II_1 holds in the following greater generality. Let V be a complex vector space and W some set of linear functionals on V. Define a function u with values in V to be analytic if $\phi \circ u$ is analytic for all ϕ in W. Then Theorem II_1 holds for analytic functions with values in V.

We now prove I and II.

THEOREM 8. *Let L be an analytic subset of a Stein manifold K. Let L have dimension d at each of its points. Then*

I. *For each p_0 in $K - L$ there exists an analytic function h on K, defined by (*) and (**) below, which vanishes on L but does not vanish at p_0.*

II. *If L is a submanifold of K then corresponding to every analytic function u on L there exists an analytic function $\tilde{u}$ on K with $\tilde{u}(q) = u(q)$ for all q in L. The function $\tilde{u}$ is given by a conglomeration of formulas which are summarized by * below.*

Proof. By Theorem 5, there exists an analytic map $f = (f_1, \cdots, f_d)$ of K into C^d whose restriction $\tilde{f}$ to L is almost proper. If $p \in K$ then $\tilde{f}^{-1}(f(p))$ is countable and consists of isolated points. Let $p_1, \cdots, p_n, \cdots$ be the points of $\tilde{f}^{-1}(f(p))$, each being counted a number of times equal to its multiplicity for the mapping $\tilde{f}$. The multiplicity of a point q for the mapping $\tilde{f}$ is defined by considering a special analytic polyhedron Q in L containing q defined by multiples of $f_1, \cdots, f_d$ and taking the multiplicity of q for the analytic polyhedron Q; this multiplicity does not depend on the choice of Q.

Let $\{S_k\}$ be a sequence of compact subsets of L, each of which has finitely many components, with $S_k \subset S_{k+1}$ for all k and $\cup_k \operatorname{int} S_k = L$. For each positive integer k let H_k be the union of those components of $\tilde{f}^{-1}(\alpha(k) \bar{E}^d)$ which intersect S_k, where $\alpha(k) > k + 1$ is chosen so that $f(S_k) \subset \alpha(k) E^d$. Thus $S_k \subset H_k$. Write

$$G_k = (\tilde{f}^{-1}(\alpha(k) \bar{E}^d) \cap H_{k+1}) - H_k$$

for all positive integers k. Now $\operatorname{int} H_k = H_k \cap \tilde{f}^{-1}(\alpha(k) E^d)$ is an analytic polyhedron defined by the functions $\alpha(k)^{-1} f_1, \cdots, \alpha(k)^{-1} f_d$. Let λ_k be the

multiplicity of $\operatorname{int} H_k$. Thus for each p in K the number of points in the sequence $p_1, p_2, \cdots$ belonging to $\operatorname{int} H_k$ is either λ_k or 0, depending on whether $f(p)$ is in $\alpha(k)E^d$ or not. By Theorem 6, there exists f_{d+1} in $A(K)$ such that

$$|f_{d+1}(p) - (\lambda_{k+1} + k^2 + 1)| < 1$$

for all p in G_k and all k. Therefore

$$\mathcal{R}(f_{d+1}(p)) \geqq \lambda_{k+1} + k^2$$

for all p in G_k. If p_0 is any point in $K - L$, we may choose f_{d+1} so that in addition

$$\exp(f_{d+1}(p_0)) \neq \exp(f_{d+1}(q))$$

for all q in $\tilde{f}^{-1}(f(p_0))$. Write

$$w = \exp(-f_{d+1})$$

so that $w \in A(K)$ and

$$|w(p)| \leqq \exp(-\lambda_{k+1} - k^2) \leqq (k^2\lambda_{k+1})^{-1}$$

for all p in G_k. Choose $k_0 \geqq 2$ with $f(T) \subset k_0 E^d$, where T is a compact subset of K to be fixed later. Since $\{H_k\}$ is an increasing sequence of sets whose union is K, there exists for each q in $\tilde{f}^{-1}(f(T))$ a smallest k with $q \in H_k$. If $k > k_0$ then $q \in G_{k-1}$; in fact, $q \in \operatorname{int} G_{k-1}$ because $f(q) \in k_0 E^d \subset kE^d$ so that $q \in \tilde{f}^{-1}(kE^d) \cap G_{k-1} \subset \operatorname{int} G_{k-1}$. If $k = k_0$ then similarly $q \in \operatorname{int} H_{k_0}$. Thus

$$\tilde{f}^{-1}(f(T)) \subset \operatorname{int} H_{k_0} \cup \bigcup_{k=k_0}^{\infty} \operatorname{int} G_k.$$

Now for each p in K and $k \geqq 1$ define

$$(*) \qquad h_k(p) = \prod (1 - w(p_i)[w(p)]^{-1}),$$

where the product is taken over all p_i in F_k, with $F_1 = H_1$ and $F_k = H_k - H_{k-1}$ for $k > 1$. Since at most λ_k of the p_i belong to H_k and since w does not vanish on K, we see that the functions h_k are well-defined. We shall show that the product

$$(**) \qquad h(p) = \prod_{k=1}^{\infty} h_k(p)$$

converges uniformly on all compact subsets of K and defines a function h in $A(K)$. Assuming for the moment that this is so, it follows from the fact that $p = p_i$ for some i whenever $p \in L$ that h vanishes on L. It follows from the fact that w has values none of which equal $w(p_0)$ at the points of $\tilde{f}^{-1}(f(p_0))$ that $h(p_0) \neq 0$. Thus h has all of the desired properties. It remains to

prove the convergence of the product $(^*_*)$ to a function h in $A(K)$. For this purpose it is sufficient to consider an open subset U of K whose closure T is compact. Fix T to be the compact set considered above. Write

$$h_0(p) = \prod (1 - w(p_i)[w(p)]^{-1}),$$

where the product is taken over all p_i in $\operatorname{int} H_{k_0}$. Thus by the above we see that for all p in T

$$h_0(p) = \prod_{k=1}^{k_0} h_k(p).$$

Similarly

$$h_k(p) = \prod (1 - w(p_i)[w(p)]^{-1})$$

for all p in T and $k \geqq k_0$, where the product is taken over all p_i in $\operatorname{int} G_k$. Now for $k \geqq k_0$, h_k is a polynomial in $[w(p)]^{-1}$ whose coefficients we denote by $a_j(p)$. Clearly a_j is an analytic function of p for all p for which the points p_i in G_k are distinct. Since $\operatorname{int} G_k$ is an analytic polyhedron defined by a multiple of the functions $f_1, \cdots, f_d$, it follows from Theorem 3 that h_k is analytic on $f^{-1}(f(\operatorname{int} G_k)) \supset U$. Similarly h_0 is analytic on U. Thus we need only check the uniform convergence of the product

$$\prod_{k=k_0}^{\infty} h_k$$

on U. To do this we may equally well consider the series

$$\sum_{k=k_0}^{\infty} (\Sigma\{p_i \text{ in } G_k : |w(p_i)|\}),$$

since $|w|$ is bounded below on the compact set T. The sum over p_i in G_k contains at most λ_{k+1} terms, each bounded by $(k^2\lambda_{k+1})^{-1}$, and so is at most k^{-2}. Thus the series converges absolutely and uniformly on T. This completes the proof of I.

We turn to the proof of II. Define $\{S_k\}$, $f_1, \cdots, f_d$, $\bar{f}$, G_k, H_k, T, p_1, $p_2, \cdots, \lambda_k$, f_{d+1}, and w as above. For p in L choose $p_1 = p$. In addition choose $f_1, \cdots, f_d$ to be a coordinate system on L at $\bar{p}$ for all $\bar{p}$ in a given 0-dimensional subset of L. Also choose w to have distinct values at the points of $\bar{f}^{-1}(f(\bar{p}))$. For each positive integer k define the function h_k on K by

$$h_k(p) = \Sigma u(p_i) c(p_i) d(p_i) [\prod_{j \neq i} (1 - w(p_j)[w(p)]^{-1})]$$

where the product and sum are taken over points p_i and p_j in H_k. Here c is a given function in $A(L)$ and d is a function in $A(K)$ which will be deter-

mined later. Let $U \subset K$ have compact closure T. Consider k so large that $f(T) \subset a(k)E^d$. Now if $f(p) \in a(k)E^d$ then $f(p_i) = f(p) \in a(k)E^d$ for all i. Thus we see that $p_i \in \operatorname{int} H_k$ whenever $f(p) \in a(k)E^d$ and $p_i \in H_k$. From this it is clear that each of the coefficients $a_j(p)$ in the expansion of $h_k(p)$ in powers of $[w(p)]^{-1}$ defines a continuous function a_j on $f^{-1}(a(k)E^d)$ which is analytic at every point p of $f^{-1}(a(k)E^d)$ which has the property that the points p_i which lie in H_k are distinct. By Theorem 3, a_j is analytic on $U \subset f^{-1}(a(k)E^d)$. Thus h_k is analytic on U for all k sufficiently large. If $j > k$ then

$$|h_j(p) - h_k(p)| \leqq AB + CD(E-1),$$

where A is the sum

$$\Sigma = \Sigma |u(p_i)||c(p_i)||d(p_i)|$$

taken over p_i in $L - H_k$, B is the product

$$\Pi = \Pi(1 + |w(p_i)||w(p)|^{-1})$$

taken over all p_i, C is the sum Σ taken over p_i in H_k, D is the product Π taken over p_i in H_k and E is the product Π taken over p_i in $L - H_k$. Thus to show uniform convergence of the sequence $\{h_k\}$ on the set U we must show that the terms AB and $CD(E-1)$ converge to 0 as $k \to \infty$. We consider only the term AB, since the treatment of the other term is entirely similar. Now the product B is finite and bounded independently of p if $p \in U$, as may be seen by considering the sum

$$\sum_{j=1}^{\infty} |w(p_j)|,$$

where the factors $|w(p)|^{-1}$ have been left out because $|w|$ is bounded independently of p in U. Thus to show that the sequence $\{h_k\}$ converges uniformly to 0 on U as $k \to \infty$ we need only show that $A \to 0$ uniformly on U as $k \to \infty$. This will be done by an appropriate choice of the function d. Let

$$\beta_k = \sup\{|u(p)c(p)| : p \in G_k\}.$$

By Theorem 5 there exists f'_{d+1} in $A(K)$ such that

$$|f'_{d+1}(p) - k^2\beta_k\lambda_{k+1} - 1| < 1$$

for all p in G_k, so that $\Re(f'_{d+1}(p)) \geqq k^2\beta_k\lambda_{k+1}$. Let $d = \exp(-f'_{d+1})$, so that $d \in A(K)$ and d vanishes nowhere on K. Also

$$|d(p)| < (k^2\beta_k\lambda_{k+1})^{-1}$$

for all p in G_k. Now if $p \in U$ and k_0 is chosen so large that $f(T) \subset k_0E^d$

each p_i in $L - H_{k_0}$ is in some G_k for $k > k_0$. Therefore the sum A splits into partial sums A_k taken over the sets G_k for $k \geqq k_0$. Now at most λ_{k+1} of the points p_i lie in G_k. It follows from the choice of d that

$$A_k \leqq (k^2\beta_k\lambda_{k+1})^{-1}\beta_k\lambda_{k+1} = k^{-2}.$$

Thus $A \leqq \sum_{k=k_0}^{\infty} k^{-2}$. Hence $\{h_k\}$ converges uniformly on U. It follows that $\{h_k\}$ converges uniformly on compact subsets of K to a function $\tilde{u}$ in $A(K)$. For p in L we have, remembering that in this case p_1 is chosen to be p,

$$h_k(p) = u(p)c(p)d(p)\prod_{j\neq 1}(1 - w(p_j)[w(p)]^{-1}),$$

where the product is taken over all p_j in H_k. Thus, passing to the limit,

$$\tilde{u}(p) = u(p)c(p)d(p)\tilde{w}(p),$$

where

$$w(p) = \prod_{j=2}^{\infty}(1 - w(p_j)[w(p)]^{-1}).$$

By the choice of $f_1, \cdots, f_d$, $p_j \neq p$ for $j \geqq 2$ if $p = \bar{p}$. By the choice of w it follows that $\tilde{w}(\bar{p}) \neq 0$.

We have thus found a function $\tilde{w} = \tilde{w}_1$ on L such that for any c_1 in $A(L)$ there exists d_1 in $A(K)$ vanishing nowhere such that there is a function $\tilde{u}_1$ in $A(K)$ which equals $uc_1d_1\tilde{w}_1$ on L. We may choose $\tilde{w}_1$ not to vanish at a 0-dimensional subset of L which contains points in each component of L. Thus the set L_1 of zeros of $\tilde{w}_1$ is of dimension at most $d-1$. Now choose $\tilde{w}_2$ to have the same properties as $\tilde{w}_1$ and to be non zero at a 0-dimensional subset of L_1 containing points in each analytic component of L_1. Continuing in this fashion we find $\tilde{w}_1, \cdots, \tilde{w}_d$ having the above properties and having no common zero on L. By a theorem of Arens [1] [2] it follows that there exist $c_1, \cdots, c_d$ in $A(L)$ with

$$\sum c_i\tilde{w}_i = 1.$$

By the choice of w_i, there exists d_i in $A(K)$ vanishing nowhere on $A(K)$ such that $uc_id_i\tilde{w}_i$ has an extension from L to an analytic function $\tilde{u}_i$ on K. Write

$$\tilde{u} = \sum_{i=1}^{d}(d_i)^{-1}\tilde{u}_i,$$

[2] The possibility of using Arens' theorem here was pointed out by Kenneth Hoffman.

so that $\tilde{u} \in A(K)$. For p in L we have

$$\tilde{u}(p) = \sum_{i=1}^{d} (d_i(p))^{-1} u(p) c_i(p) d_i(p) \tilde{w}_i(p)$$
$$= u(p) \sum_{i=1}^{d} c_i(p) \tilde{w}_i(p) = u(p).$$

This completes the proof of II. The final formula reads

$$* \qquad \tilde{u}(p) = \sum_{n=1}^{d} (d_n(p))^{-1} \sum_{i=1}^{\infty} u(p_i) c_n(p_i) d_n(p_i) [\prod_{j \neq i} (1 - w_n(p_j)[w_n(p)]^{-1})].$$

Remark. As is well-known, II (in fact II_1) can be used to prove Runge's theorem for Stein manifolds. The proof goes as follows. Let T be a compact subset of a Stein manifold K and h a function analytic in some neighborhood U of T. Runge's theorem says that h can be uniformly approximated on T by functions in $A(K)$ if for each p in $U - T$ there exists in $A(K)$ with $|g(p)| > \max\{|g(q)| : q \in T\}$. To show this notice that the assumption implies that there exists an analytic polyhedron P in K with

$$T \subset P \subset U.$$

Let $f_1, \cdots, f_k$ be the functions defining P. Choose $f_{k+1}, \cdots, f_n$ in $A(K)$ to separate points of U and to have rank equal to the dimension of K at each point of U. We may assume that $|f_i(p)| < 1$ for all p in T, $k+1 \leqq i \leqq n$. Thus the map $f = (f_1, \cdots, f_n)$ of K into C^n maps T into E^n. Also $M = f(P)$ is a submanifold of E^n. Let $\hat{f}$ be the analytic homeomorphism of M onto P which is inverse to f, and let u be the analytic function $h \circ \hat{f}$ on L. By II, u admits an analytic extension $\tilde{u}$ to E^n. By a well-known and elementary use of the Cauchy integral formula, $\tilde{u}$ can be uniformly approximated on $f(T)$ by functions v analytic on E^n. Thus the functions $v \circ f$ in $A(K)$ uniformly approximate h on T, as was desired.

4. The resolution of Cousin distributions. As is well-known, the solution on a Stein manifold of Cousin's first problem, which consists of the global construction of a meromorphic function whose principal parts are locally given, can be obtained from the fact that on a Stein manifold the sheaf of germs of analytic functions has a first cohomology group which is trivial. Cousin's second problem, which states that a globally defined meromorphic function exists whose quotients with certain locally given meromorphic functions are analytic functions without zeros, can also be solved using this statement about cohomology. Thus we shall restrict ourselves in this section to showing that the cohomology group in question is trivial. This will be done without the use of cohomological terminology.

Definition 5. Let U be an open subset of a Stein manifold K. A *Cousin distribution* on U will consist of a covering $\{U_\alpha\}$ of U by open sets and an assignment for each α and β of an analytic function $f_{\alpha\beta}$ to the intersection $U_\alpha \cap U_\beta$ such that

(a) $$f_{\alpha\beta} + f_{\beta\alpha} = 0$$

and

(b) $$f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0 \qquad \text{on } U_\alpha \cap U_\beta \cap U_\gamma.$$

for all α, β, and γ. By a *resolution* of the given distribution $\{U_\alpha\}$, $\{f_{\alpha\beta}\}$ on an open set $S \subset K$ we mean a family $\{f_\alpha\}$, where f_α is analytic on $S \cap U_\alpha$, such that

$$f_{\alpha\beta} = f_\alpha - f_\beta \qquad \text{on } S \cap U_\alpha \cap U$$

for all α and β.

By a *refinement* of the distribution $\{U_\alpha\}$, $\{f_{\alpha\beta}\}$ we mean a Cousin distribution $\{V_\gamma\}$, $\{g_{\gamma\delta}\}$ and a mapping σ of the index set of $\{V_\gamma\}$ into the index set of $\{U_\alpha\}$ such that

(c) $$V_\gamma \subset U_{\sigma(\gamma)}$$

and

(d) $$g_{\gamma\delta} = f_{\sigma(\gamma)\sigma(\delta)} \qquad \text{on } V_\gamma \cap V_\delta$$

for all γ and δ. If T is a compact subset of K then a Cousin distribution on T means a Cousin distribution on some neighborhood U of T, and a resolution on T means a resolution on some neighborhood of T. The same definitions will be taken to hold if the functions $f_{\alpha\beta}$ etcetera are replaced by polynomials of given degree in an indeterminate X whose coefficients are analytic functions on the appropriate sets.

We make use below of the trivial fact that a Cousin distribution which has a refinement with a resolution itself has a resolution.

The cohomology theorem mentioned above is just the statement that every set of Cousin data on a Stein manifold K has a resolution on K, and it is this fact which will be proved below. As a tool we shall use the fact that every Cousin distribution on the subset E^n of C^n has a resolution. This is well-known to be a completely elementary fact and indeed was essentially proved by Cousin [4]. For completeness however we sketch the proof at the end of this section. The following lemma contains the essence of the theorem to be proved.

LEMMA 1. *Let T be a compact set of a Stein manifold K and Δ a neighborhood of T such that for all p in $\Delta - T$ there exists f in $A(K)$ with*

$$f(p) > \sup\{|f(q)| : q \in T\}.$$

Then any Cousin distribution on T has a resolution on the interior M of T.

Proof. Let the given Cousin data $\{U_\alpha\}$, $\{f_{\alpha\beta}\}$ be defined on an open set $U \supset T$. Let P be a special analytic polyhedron defined by functions $f_1, \cdots, f_d$ in $A(K)$ with $T \subset P \subset \bar{P} \subset U$. Choose a covering $\{V_\gamma\}$ of E^d so fine that each component of $W_\gamma = P \cap f^{-1}(V_\gamma)$ lies in one of the sets U_α for each γ. Since P is a special analytic polyhedron, each W_γ has only a finite number of components, which we denote by $W_1^\gamma, \cdots, W_t^\gamma$, where t depends on γ. For each of these components W_i^γ choose $U_{\alpha(i\gamma)}$ in $\{U_\alpha\}$ with $W_i^\gamma \subset U_{\alpha(i\gamma)}$. For each γ and δ define the analytic function $h_{\gamma\delta}$ on $W_\gamma \cap W_\delta$ by

$$h_{\gamma\delta}(p) = f_{\alpha(i\gamma)\alpha(j\delta)}$$

for all p in $W_i^\gamma \cap W_j^\delta$. It is easy to see that $\{W_\gamma\}$, $\{h_{\gamma\delta}\}$ is a Cousin distribution on P. For each point z in E^d let $p_1, \cdots, p_\lambda$ be the points of $\omega(z) = \{p_1, \cdots, p_\lambda\}$. Let w be any function in $A(K)$. Let X be an indeterminate. For each γ and δ and each z in $V_\gamma \cap V_\delta$ define

$$g_{\gamma\delta}(z, X) = \sum_{j=1}^{\lambda} h_{\gamma\delta}(p_i) \prod_{j \neq i} (X - w(p_j)).$$

By Theorem 3 we see that the coefficient of each power of X in the polynomial $g_{\gamma\delta}$ in X is analytic on $V_\gamma \cap V_\delta$. Since $\{W_\gamma\}$, $\{h_{\gamma\delta}\}$ is a Cousin distribution on P we see that

$$g_{\gamma\delta} + g_{\delta\gamma} = 0$$

and

$$g_{\gamma\delta} + g_{\gamma\sigma} + g_{\sigma\gamma} = 0$$

for all γ, δ, and σ. Thus $\{V_\gamma\}$, $\{g_{\gamma\delta}\}$ is a Cousin distribution for E^d. By the statement made above (whose proof will be outlined below), there exists a resolution $\{g_\gamma\}$ of this Cousin distribution. Now $\{W_i^\gamma\}$ is a covering of P. For each W_i^γ define the function $F_{i\gamma}$ an W_i^γ by

$$F_{i\gamma}(p) = g_\gamma(f(p), w(p)).$$

These functions are the resolution of a Cousin distribution $\{W_i^\gamma\}$, $\{F_{i\gamma j\delta}\}$ on P, where $F_{i\gamma j\delta}$ is the function

$$F_{i\gamma}(p) - F_{j\delta}(p) = g_{\gamma\delta}(f(p), w(p))$$

on $W_i^\gamma \cap W_j^\delta$. If for each p in P we write $\omega(f(p)) = \{p_1, \cdots, p_\lambda\}$, with $p_1 = p$, it follows that

$$\begin{aligned} F_{i\gamma j\delta}(p) &= \sum_{i=1}^{\lambda} h_{\gamma\delta}(p_i) \prod_{j \neq i} (w(p) - w(p_j)) \\ &= h_{\gamma\delta}(p) \prod_{i=2}^{\lambda} (w(p) - w(p_j)) \\ &= \tilde{w}(p) h_{\gamma\delta}(p) = \tilde{w}(p) f_{\alpha(i\gamma)\alpha(j\delta)}(p), \end{aligned}$$

where $\tilde{w}$ is the analytic function on P defined by

$$\tilde{w}(p) = \prod_{j=2}^{\lambda} (w(p) - w(p_j)).$$

Thus $\{F_{i\gamma j\delta}\}$ is a resolution on P, and therefore on M, of the distribution $\{W_i^\gamma\}$, $\{\tilde{w} f_{\alpha(i\gamma)\alpha(j\delta)}\}$. Since this distribution is a refinement on M of the distribution $\{U_\alpha\}$, $\{\tilde{w} f_{\alpha\beta}\}$, it follows that the latter distribution has a resolution on M.

If p_0 is any point in T we may assume that $f_1, \cdots, f_d$ were chosen so that f has multiplicity 1 at p_0 and that w was chosen to distinguish p_0 from the other points of $\omega(f(p_0))$, so that $\tilde{w}(p_0) \neq 0$. Thus we can find a finite number of functions $\tilde{w}_1, \cdots, \tilde{w}_n$, each analytic in some neighborhood of T and having no common zero on T, such that for $1 \leqq i \leqq n$ the distribution $\{U_\alpha\}$, $\{\tilde{w}_i f_{\alpha\beta}\}$ has a resolution on M. Thus for each c_i analytic on M the distribution $\{U_\alpha\}$, $\{\tilde{w}_i c_i f_{\alpha\beta}\}$ has a resolution on M. By Theorem 4, the c_i may be chosen ith $\sum c_i \tilde{w}_i = 1$. It follows that distribution $\{U_\alpha\}$, $\{f_{\alpha\beta}\}$, which on M is therefore the sum of distributions which have resolutions on M, has a resolution on M. This completes the proof.

THEOREM 9. *Any Cousin distribution on a Stein manifold K has a resolution on K.*

Proof. Since K is Stein, there exists an increasing sequence $\{M_n\}$ of open sets whose union is K and whose closures T_n all have the property of the set T of Lemma 1. Thus for each n by Lemma 1 the given Cousin distribution $\{U_\alpha\}$, $\{f_{\alpha\beta}\}$ has a resolution $\{f_\alpha{}^n\}$ on M_n. We may assume that the closure of each U_α is compact. Let $\{S_n\}$ be an increasing sequence of compact sets whose union is K, which have the property of the set T in Lemma 1, and which satisfy the inclusion $S_n \subset M_n$. We show now by induction on n that for each n there exists a resolution $\{h_\alpha{}^n\}$ of $\{U_\alpha\}$, $\{f_{\alpha\beta}\}$ on M_n such that

$$|h_\alpha{}^{n-1}(p) - h_\alpha{}^n(p)| < 2^{-n}$$

for all p in S_{n-1}. Assume that the $h_\alpha{}^n$ have been found for all integers

smaller than a given integer n. For each p in M_{n-1} choose α with p in U_α and write

$$g_0(p) = h_\alpha{}^{n-1}(p) - f_\alpha{}^n(p).$$

It is easily seen that g_0 is an analytic function on M_{n-1}. By Runge's Theorem there exists g in $A(K)$ with

$$|g(p) - g_0(p)| < 2^{-n}$$

for all p in S_{n-1}. It follows that the functions $\{h_\alpha{}^n\}$ have the required properties, where $h_\alpha{}^n = f_\alpha{}^n + g$ on $U_\alpha \cap M_n$. Since the closure of each U_α is compact, it follows that for each fixed α the sequence $\{h_\alpha{}^n\}$ converges uniformly on U_α to a function $\{f_\alpha\}$. These functions clearly form a resolution on K of the given Cousin distribution. This completes the proof.

Remark. If a meromorphic function with values in a Banach space is defined to be one which is locally a quotient of an analytic Banach-space valued function by an analytic scalar-valued function, then Cousin's first and second problems can be solved for such functions by applying Theorem 9 for Banach-space valued Cousin distributions.

To finish we indicate the technique for obtaning a resolution of a Cousin distribution on E. Let D be the set

$$\{(z_1, \cdots, z_d) \in C^d : z = x_n + iy_n, |x_n| < 1, |y_n| < 1, 1 \leqq n \leqq d\}.$$

Since D^d is conformally equivalent to E^d, we may equally well consider the resolution of a Cousin distribution given on D^d. In fact we may assume that the distribution is given on the closure H^d of D^d, as the proof of Theorem 9 shows. Let the given distribution be $\{U_\alpha\}$, $\{f_{\alpha\beta}\}$, given on H^d. We first introduce the notion of a *soft resolution* for a Cousin distribution. This is the same as a resolution except that the functions f_α are permitted to be the arbitrary (not necessarily analytic) functions. It is easy to see that every Cousin distribution has a soft resolution. Let $\{F\}_\alpha$ be a soft resolution of the given distribution. If U is an open set then it is easy to check that the given distribution has a resolution on U if and only if there exist a function F on U such that $F - F_\alpha$ is analytic on $U_\alpha \cap U$ for all α. Let $z_n = x_n + iy_n$, $1 \leqq n \leqq d$, be the coordinates in C^d. Let N be a positive integer and write

$$S_n(k) = \{z \in H^d : -1 + \frac{k-1}{N} \leqq x_n \leqq -1 + \frac{k}{N}\}$$

for $1 \leqq k \leqq 2N$ and $1 \leqq n \leqq 2d$. For each choice of $k_1, \cdots, k_{2d}$, with $1 \leqq k_j \leqq 2N$, write

$$S(k_1, \cdots, k_{2d}) = S_1(k_1) \cap \cdots \cap S_{2d}(k_{2d}).$$

Choose N so large that the covering $\{S(k_1,\cdots,k_{2d})\}$ of H^d is a refinement of the covering $\{U_\alpha\}$ of H^d. For each $k_1,\cdots,k_t$, with $1\leqq k_j\leqq 2N$ and $0\leqq t\leqq 2d$, let

$$S(k_1,\cdots,k_t)=S_1(k_1)\cap\cdots\cap S_t(k_t)$$

so that $S(\emptyset)=H^d$. We shall show that the given Cousin distribution has a resolution on each $S(k_1,\cdots,k_t)$. This is true if $t=2d$ because the covering $\{S(k_1,\cdots,k_{2d})\}$ is a refinement of $\{U_\alpha\}$. Assume therefore that it is true for all t greater than some given t, so that we must show that there is a resolution on each $S(k_1,\cdots,k_t)$. To this end write for each $1\leqq k\leqq 2N$

$$S(k_1,\cdots,k_t;k)=S(k_1,\cdots,k_t)\cap\bigcup_{j=1}^{k}S_{t+1}(j)=\bigcup_{j=1}^{k}S(k_1,\cdots,k_t,j)$$

Thus

$$S(k_1,\cdots,k_t;k+1)=S(k_1,\cdots,k_t;k)\cup S(k_1,\cdots,k_t,k+1)$$

and

$$S(k_1,\cdots,k_t;1)=S(k_1,\cdots,k_t,1).$$

By the induction hypothesis on t, it follows that the given distribution has a resolution on $S(k_1,\cdots,k_t;1)$. We now show by induction on k (t remaining fixed) that the given distribution has a resolution on $S(k_1,\cdots,k_t;k)$. Assume therefore that this is true for all k smaller than the given k. Thus by the above we see that there exists a neighborhood V_1 of $S(k_1,\cdots,k_t;k-1)$ and a function G_1 on V_1 such that G_1-F_α is analytic on $V_1\cap U_\alpha$ for all α. Similarly, there exists a function G_0 on some neighborhood V_2 of $S(k_1,\cdots,k_t,k)$ such that G_2-F_α is analytic on $V_2\cap U_\alpha$ for all α. Thus G_1-G_2 is analytic on $V_1\cap V_2$. It is well known and easy to see that therefore there exist open sets W_1 and W_2 with

$$S(k_1,\cdots,k_t;k-1)\subset W_1\subset V_1$$

and

$$S(k_1,\cdots,k_t,k)\subset W_2\subset V_2$$

and analytic functions A_1 and A_2 on W_1 and W_2 respectively such that A_2-A_1 and G_1-G_2 are equal on $W_1\cap W_2$. It follows that the function B on $W_1\cup W_2$ defined as A_1+G_1 on W_1 and A_2+G_2 on W_2 has the property that $B-F_\alpha$ is analytic on $(W_1\cup W_2)\cap U_\alpha$ for all α. Thus the given distribution has a resolution on

$$S(k_1,\cdots,k_t;k)=S(k_1,\cdots,k_t;k-1)\cup S(k_1,\cdots,k_t,k)\subset W_1\cup W_2.$$

This completes the induction on k. It follows that there exists a resolution on

$$S(k_1,\cdots,k_t;2N)=S(k_1,\cdots,k_t).$$

This in turn completes the induction on t. It follows that there exists a resolution on $S(\emptyset) = H^d$, as was to be proved.

Added in proof: Let L be an analytic submanifod of dimension d in a Stein manifold K. Let U be an open relatively compact subset of K. Then the contruction used in the proof of Theorem II_1 shows that there exists a multiple-valued projection of U into L in the following sense. There exists an integer λ and an analytic map

$$p \to \{(p_1, z_1), \cdots, (p_\lambda, z_\lambda)\}$$

of U into the λ-fold unordered product ${}_\lambda\Omega$ of the space $\Omega = L \times C$ with itself having the following properties. For each p in U,

$$z_1 + \cdots + z_\lambda = 1.$$

For each p in $L \cap U$ and each i with $1 \leqq i \leqq \lambda$ for which $p_i \neq p$ we have $z_i = 0$. This fact in turn clearly implies Theorem II_1, since u can be extended from L to U by the formula

$$u(p) = \sum_{i=1}^{\lambda} z_i u(p_i).$$

Similarly the proof of Theorem II can be used to show the existence of certain multiple-valued analytic projections of K into L.

UNIVERSITY OF CALIFORNIA, BERKELEY.

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ON INDUCED REPRESENTATIONS II: INFINITESIMAL INDUCTION.*

By ROBERT J. BLATTNER.[1]

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1. Introduction. This paper is a continuation of the investigation begun in [1].[2] We take as our point of departure some observation of Gelfand and Graev in [3] concerning the appearance of analyticity restrictions on the functions upon which the non-degenerate principal series of representations of $SL(n, \boldsymbol{R})$ act. Their argument runs as follows: Let $K(n, \boldsymbol{C})$ be the subgroup of $SL(n, \boldsymbol{C})$ consisting of the superdiagonal matrices of determinant unity. The non-degenerate principal series of representations of $SL(n, \boldsymbol{C})$ is realized in a space of functions defined on the right coset space $SL(n, \boldsymbol{C})/K(n, \boldsymbol{C})$, or alternately in a space of functions defined on $SL(n, \boldsymbol{C})$ which are (left) invariant under the action of $K(n, \boldsymbol{C})$. Now the non-degenerate principal series of $SL(n, \boldsymbol{R})$ may be thought of as acting in spaces of functions on $SL(n, \boldsymbol{R})$ which are invariant under the (left) action of a subgroup of $SL(n, \boldsymbol{C})$ of the form $xK(n, \boldsymbol{C})x^{-1}$, $x \in SL(n, \boldsymbol{C})$. This statement is to be interpreted as honest invariance under $SL(n, \boldsymbol{R}) \cap xK(n, \boldsymbol{C})x^{-1}$ together with differential left invariance under the Lie algebra of $xK(n, \boldsymbol{C})x^{-1}$, regarded as a complex subalgebra of the complexification of the Lie algebra of $SL(n, \boldsymbol{R})$. If the functions are taken to be defined on

$$\mathcal{M} = SL(n, \boldsymbol{R})/[SL(n, \boldsymbol{R}) \cap xK(n, \boldsymbol{C})x^{-1}],$$

then invariance under the action of the Lie algebra goes over into analyticity on certain complex submanifolds of $\mathcal{M}$.

Now it is known that the non-degenerate principal series of $SL(n, \boldsymbol{C})$ consists exactly of the representations of $SL(n, \boldsymbol{C})$ induced by characters of $K(n, \boldsymbol{C})$ in the sense of Mackey [5]. It is therefore natural to ask for a generalization of the notion of induced representation which for the case at hand gives the non-degenerate principal series of $SL(n, \boldsymbol{R})$. Our generalization has as its setting a Lie group G with Lie algebra $\mathfrak{g}$, a complex subalgebra $\mathfrak{u}$ of the complexification $\mathfrak{g}_c$ of $\mathfrak{g}$, a closed subgroup Γ of G whose Lie algebra

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

is $\mathfrak{g} \cap \mathfrak{n}$, a unitary representation L of Γ on a finite dimensional Hilbert space $\mathfrak{V}$, and a representation ρ of $\mathfrak{n}$ on $\mathfrak{V}$. Certain restrictions are placed on $\mathfrak{n}$ and ρ (for exact statements, see Section 3). The corresponding induced representation $U^{L,\rho}$ is a subrepresentation of the representation U^L (see [1] for the definition). Here too analyticity restrictions appear on the functions upon which $U^{L,\rho}$ operates (Proposition 2). Finally, the representations of Gelfand and Graev are of the form we propose, except for certain degenerate cases.

Section 2 develops the necessary preliminary material on manifolds with some real and some complex parameters. Theorem 1 gives us a way of producing such manifolds. In Section 3, we introduce our induced representations. The problem of calculating intertwining numbers is the subject of Section 4. The basic result is Theorem 3 and is based on the intertwining number theorem in [1]. (Applications of Theorem 3 to the various series of representations of the real semi-simple groups is reserved to another paper.) In Section 5, additional restrictions on the complex subalgebras involved allows us to reduce the intertwining number problem to one of solving certain differential equations, while Theorem 4 gives us effective irreducibility and disjointness criteria for the representations arising from this restricted class of subalgebras. Finally, in Section 6, we consider two examples:

(a) Harish-Chandra has also considered the problem of generalizing the notion of induced representation in case G is a semisimple Lie group and Γ is a special sort of Cartan subgroup. His results may be found in [4]. Our irreducibility and inequivalence criteria are applied to his representations.

(b) In [3], Gelfand and Graev mention a series of representations of $SL(2k, \boldsymbol{R})$ which are not in the non-degenerate principal series. Once again our irreducibility and inequivalence criteria apply.

2. Partially complex manifolds. We begin by laying down some conventions and recalling some facts.

Vector spaces. If $\mathfrak{W}$ is a complex vector space, then $\mathfrak{W}_r$ will denote the real vector space obtained from $\mathfrak{W}$ by restricting the field of scalars. If $\mathfrak{W}$ is a real vector space, then $\mathfrak{W}_c$ is the complexification of $\mathfrak{W}$, and $\mathfrak{W}$ will be regarded as a real subspace of $\mathfrak{W}_{cr}$. In $\mathfrak{W}_c$, $^+$ will denote the unique conjugate linear isomorphism which $= -I$ on $\mathfrak{W}$. if Ψ is a linear map of the real vector space $\mathfrak{V}$ into the real vector space $\mathfrak{W}$, then Ψ_c will denote the usual extension of Ψ to a complex linear map of $\mathfrak{V}_c$ into $\mathfrak{W}_c$. If $\mathfrak{W}$ is a complex vector space, then κ is the unique complex linear map of $\mathfrak{W}_{rc}$ onto $\mathfrak{W}$

which leaves every point of $\mathfrak{W}_r = \mathfrak{W}$ fixed. If $\mathfrak{V}$ and $\mathfrak{W}$ are complex vector spaces and if Ψ is a linear map from $\mathfrak{V}_r$ into $\mathfrak{W}_r$, then Ψ is a complex linear if and only if Ψ_c carries $\ker(\kappa)$ into $\ker(\kappa)$.

Differentiable Manifolds. All functions and vector fields will be assumed to have open domains unless otherwise stated. Let $\mathcal{M}$ be a manifold. If $p \in \mathcal{M}$, then $\mathcal{M}_p$ will denote the tangent space to $\mathcal{M}$ at p. If f is a differentiable function from $\mathcal{M}$ to a complex vector space, if $p \in \mathrm{dom}(f)$, and if $X_p \in \mathcal{M}_{pc}$, then $X_p f$ is defined in the obvious way. A *complex distribution* of dimension r on $\mathcal{M}$ is a function $\mathfrak{m}$ from $\mathcal{M}$ to $\bigcup_{p \in \mathcal{M}} \mathcal{M}_{pc}$ such that $\mathfrak{m}_p$ (the value of $\mathfrak{m}$ at p) is an r-dimensional complex subspace of $\mathcal{M}_{pc}$; the properties of being analytic, involutive, etc., are defined in the obvious way. If ϕ is a differentiable map of $\mathcal{M}$ into another differentiable manifold, then ϕ_* will denote the induced map of tangent spaces.

Complex manifolds. Let $\mathcal{M}$ be equipped with a complex structure. Then $\mathcal{M}_p{}^c$ will denote $\mathcal{M}_p$ equipped with the canonically induced complex linear structure. To each $p \in \mathcal{M}$ we assign $\mathfrak{c}$, the kernel of κ in $\mathcal{M}_{pc}$ $(= \mathcal{M}_p{}^c{}_{rc})$. $\mathfrak{c}$ is a complex C^ω involutive distribution called the *canonical distribution* of $\mathcal{M}$. If ϕ is a differentiable map of $\mathcal{M}$ into another complex manifold, then ϕ is called *holomorphic* if ϕ_* is complex linear. This is equivalent to demanding that ϕ_{*c} take $\mathfrak{c}_p$ into $\mathfrak{c}_{\phi(p)}$ for all $p \in \mathrm{dom}(\phi)$. If f is a function from $\mathcal{M}$ to a complex vector space, then f is holomorphic in the above sense if and only if $\mathfrak{c}_p f = 0$ for all $p \in \mathrm{dom}(f)$ (Cauchy-Riemann equations). If X is a vector field with values in $\mathcal{M}_{pc}$, $p \in \mathcal{M}$, then X will be called *holomorphic* if Xf is holomorphic for all holomorphic functions with values in $\boldsymbol{C}$.

Mixed spaces. The canonical projection of $\boldsymbol{C}^m \times \boldsymbol{R}^n$ onto $\boldsymbol{C}^m$ (resp. $\boldsymbol{R}^n$) will be denoted θ (resp. ρ). If $x \in R^n$, then θ_x will denote the map $z \to (z, x)$ of $\boldsymbol{C}^m$ into $\boldsymbol{C}^m \times \boldsymbol{R}^n$.

Our basic notion, that of a partially complex structure on a differentiable manifold, is to the mixed space $\boldsymbol{C}^m \times \boldsymbol{R}^n$ as the notion of complex structure is to the space $\boldsymbol{C}^m$.

Definition. Let $\mathcal{M}$ be a real C^ω manifold of dimension n. A partially complex structure of dimension r $(2r \leqq n)$ on $\mathcal{M}$ is a maximal collection $\mathcal{B}$ of maps satisfying:

(1) each $\phi \in \mathcal{B}$ is a C^ω diffeomorphism of an open set in $\mathcal{M}$ with an open set in $\boldsymbol{C}^r \times \boldsymbol{R}^{n-2r}$;

(2) the domains of the $\phi \in \mathcal{B}$ cover $\mathcal{M}$;

(3) if $\phi, \psi \in \mathcal{B}$ and $x \in \boldsymbol{R}^{n-2r}$, then $\rho \circ \phi \circ \psi^{-1}$ depends locally only on $\boldsymbol{R}^{n-2r}$ and $\theta \circ \phi \circ \psi^{-1} \circ \theta_x$ is holomorphic.

Remark. As usual, if $\mathcal{B}_0$ is any collection satisfying (1-3), then $\mathcal{B}_0$ is contained in a unique maximal collection $\mathcal{B}$ satisfying (1-3). Hence to define such a structure, we may ignore the maximality condition.

An open set O in $\mathcal{M}$ is called *cubical* if there is a coordinate function $\phi \in \mathcal{B}$ with domain O whose image is a set of the form $[(z_1, \cdots, z_r, x_1, \cdots, x_{n-2r}) : |z_i| < a, |x_i| < a]$ for some fixed $a > 0$.

We introduce a complex distribution $\mathfrak{m}$ as follows: if $\phi \in \mathcal{B}$ and $p \in \operatorname{dom} \phi$, then $\mathfrak{m}_p = \ker[(\rho \circ \phi)_{*c}] \cap (\theta \circ \phi)_{*c}^{-1}(\mathfrak{c}_{\theta(\phi(p))})$. It follows from condition (3) above that $\mathfrak{m}_p$ is independent of the choice of ϕ. From the way $\mathfrak{m}$ looks under ϕ_{*c} for a fixed $\phi \in \mathcal{B}$, it is clear that $\mathfrak{m}$ is involutive, that $\mathfrak{m} \cap \mathfrak{m}^+ = \{0\}$, and that $\mathfrak{m} + \mathfrak{m}^+$ is involutive, being in fact the complexification of the tangent distribution to slices of the form $[p \in \operatorname{dom} \phi : \rho(\phi(p)) = \text{constant}]$. Each maximal integral manifold of $\mathfrak{m} + \mathfrak{m}^+$ is a complex manifold in the obvious fashion and will be called a *canonical complex submanifold* of $(\mathcal{M}, \mathcal{B})$.

THEOREM 1. *Let $\mathcal{M}$ be a real C^ω manifold of dimension n. Let $\mathfrak{m}$ be an involutive complex distribution $r (2r \leqq n)$ on $\mathcal{M}$ such that $\mathfrak{m} \cap \mathfrak{m}^+ = \{0\}$ and $\mathfrak{m} + \mathfrak{m}^+$ is involutive. Then $\mathcal{M}$ bears a unique partially complex structure $\mathcal{B}$ of dimension r such that $\mathfrak{m}$ is the canonical distrbution of $(\mathcal{M}, \mathcal{B})$.*

Proof. Let $p \in \mathcal{M}$. By the complex Frobenius Theorem there exists complex valued C^ω functions $f_1, \cdots, f_n$ defined in an open neighborhood N of p such that for all $q \in N$ the annihilators in $\mathcal{M}_{qc}$ of $\{f_1, \cdots, f_n\}$, $\{f_{r+1}, \cdots, f_n\}$, and $\{f_{2r+1}, \cdots, f_n\}$ are $\{0\}$, $\mathfrak{m}_q$, and $\mathfrak{m}_q + \mathfrak{m}_q^+$ respectively. Without loss of generality we may assume that N is a cubical neighborhood of breadth a about p with real coordinate functions $g_1, \cdots, g_n$ such that slices of the form $g_i(p) = \xi_i$, $|\xi_i| < a$, $i = 2r+1, \cdots, n$ are integral manifolds of $\mathfrak{m} + \mathfrak{m}^+$. Set $\phi = (f_{r+1} \times \cdots \times f_{2r}) \times (g_{2r+1} \times \cdots \times g_n)$, a C^ω map of N into $\boldsymbol{C}^r \times \boldsymbol{R}^{n-2r}$. ϕ is regular. In fact, if $\phi_x(X_q) = 0$, $X_q \in \mathcal{M}_q$, $q \in N$, then $X_q \in \mathfrak{m} + \mathfrak{m}^+$ so that $X_q f_i = 0$ for $i = r+1, \cdots, n$. Hence $X_q \in \mathfrak{m}_q \cap \mathcal{M}_q = \{0\}$. Shrinking N if necessary, we may assume that ϕ is a diffeomorphism of N with an open subset of $\boldsymbol{C}^r \times \boldsymbol{R}^{n-2r}$.

Let $\mathcal{B}_0$ be the collection of all ϕ's obtained in this fashion for arbitrary $p \in \mathcal{M}$. Properties 1 and 2 in the definition are true by construction. As for property 3, let $\phi, \psi \in \mathcal{B}_0$. Let $y \in \operatorname{dom}(\rho \circ \phi \circ \psi^{-1})$ and let N_0 be a neighbor-

hood of $\psi^{-1}(y)$ such that $\psi(N_0)$ is a cube $\subseteq \operatorname{dom}(\rho \circ \phi \circ \psi^{-1})$. Then slices of N_0 of the form $[p \in N_0 : \rho(\psi(p)) = \text{constant}]$ are integral manifolds of $\mathfrak{m} + \mathfrak{m}^+$ contained in $\operatorname{dom} \phi$. Therefore $\rho \circ \phi$ is constant on these slices, and this implies that $\rho \circ \phi \circ \psi^{-1}$ restricted to $\psi(N_0)$ depends only on $\boldsymbol{R}^{n-2r}$. Now let $x \in \boldsymbol{R}^{n-2r}$, set $h = \theta \circ \phi \circ \psi^{-1} \circ \theta_x$, and let $z \in \operatorname{dom} h$. Observe that $(\psi^{-1} \circ \theta_x)_*$ maps $\mathfrak{c}_z$ univalently onto $\mathfrak{m}_{\psi^{-1}(z,x)}$ and that $(\theta \circ \phi)_*$ maps this space in turn univalently onto $\mathfrak{c}_{h(z)}$. Since h_* maps $\mathfrak{c}_z$ into $\mathfrak{c}_{h(z)}$, we conclude that h is holomorphic.

If $\mathcal{B}$ is the partially complex structure on $\mathcal{M}$ determined by $\mathcal{B}_0$, it is clear that $\mathfrak{m}$ is the canonical distribution of $(\mathcal{M}, \mathcal{B})$. As for uniqueness, let $\mathcal{B}'$ be any partially complex structure satisfying the conclusion of the theorem and let $\phi \in \mathcal{B}_0$. The arguments of the last paragraph show that $\mathcal{B}' \cup \{\phi\}$ also satisfy the theorem. By maximality we conclude that $\mathcal{B} = \mathcal{B}'$.

3. The induced representations. For any unexplained notation concerning induced representations the reader is referred to [1]. Let G be a Lie group with left invariant Lie algebra $\mathfrak{g}$ and modular functions Δ. If $X \in \mathfrak{g}$ and $x(.)$ is the one-parameter subgroup of G lying beneath X, we have $(Xf)(y) = D_t f(yx(t))\big|_{t=0}$ for all differentiable functions f. We define a mapping $'$ of the family of differentiable functions onto itself by $f'(x) = f(x^{-1})$. Clearly $'$ is involutive. If X is a differential operator on G with C^∞ coefficients, we define $X^\square$ by $X^\square f = (Xf')'$ for all C^∞ functions f. $\square$ is clearly an involutive automorphism of the algebra of all C^∞ everywhere defined differential operators on G. If $X \in \mathfrak{g}$ and $x(.)$ lies beneath X, we have $(X^\square f)(y) = -D_t f(x(t)y)\big|_{t=0}$.

Let Γ be a closed subgroup of G with Lie algebra $\mathfrak{h}$ and modular function δ. Let $\mathfrak{n}$ be a complex subalgebra of $\mathfrak{g}_c$ such that $\mathfrak{n} \cap \mathfrak{n}^+ = \mathfrak{h}_c$, $(\operatorname{ad} \Gamma)\mathfrak{n} = \mathfrak{n}$, and $\mathfrak{n} + \mathfrak{n}^+$ is a Lie algebra. Let $\mathcal{M} = G\big|\Gamma$ and let π be the canonical projection of G onto $\mathcal{M}$.

PROPOSITION 1. *$\pi_*(\mathfrak{n}^\square{}_x)$ depends only on $\pi(x)$. Moreover, if we set $\mathfrak{m}_p = \pi_*(\mathfrak{n}^\square{}_x)$ where $\pi(x) = p$, then $\mathfrak{m}$ is the canonical distribution of a unique partially complex structure $\mathcal{B}$ on $\mathcal{M}$.*

Proof. Let $X \in \mathfrak{g}$ and $x(.)$ be the one-parameter subgroup of G lying beneath X. Let f be a differentiable function on $\mathcal{M}$. Then

$$\begin{aligned}\pi_*(X^\square{}_{\xi g})f = X^\square{}_{\xi g}(f \circ \pi) &= -D_t f(\pi(x(t)\xi g))\big|_{t=0} \\ &= -D_t f(\pi(\xi^{-1}x(t)\xi g))\big|_{t=0} = \pi_*(\operatorname{ad} \xi^{-1}X)^\square{}_g)f \quad .\end{aligned}$$

for $\xi \in \Gamma$, $g \in G$. Since $(\mathrm{ad}\,\Gamma)\mathfrak{n} = \mathfrak{n}$, we conclude the truth of our first assertion.

Since the kernel of $\pi_* | \mathfrak{n}^{\square}{}_x$ is just $\mathfrak{h}_x = \mathfrak{n}_x \cap \mathfrak{n}_x{}^+$, we see that $\mathfrak{m} \cap \mathfrak{m}^+ = \{0\}$. If α is a local cross-section of G over $\mathcal{M}$, then $\alpha_*(\mathfrak{m}_p) = \mathfrak{n}^{\square}{}_{\alpha(p)}$ and

$$\alpha_*(\mathfrak{m}_p + \mathfrak{m}_p{}^+) = \mathfrak{n}^{\square}{}_{\alpha(p)} + \mathfrak{n}^{\square}{}_{\alpha(p)^+} = (\mathfrak{n} + \mathfrak{n}^+)^{\square}{}_{\alpha(p)}$$

for all $p \in \mathrm{dom}(\alpha)$. It follows from [2], Ch. III, § 5, Proposition 2, that $\mathfrak{m}$ and $\mathfrak{m} + \mathfrak{m}^+$ are involuntary. Our second assertion now follows from Theorem 1.

Our induced representations are obtained as follows: Let L be a unitary representation of Γ on the finite dimensional Hilbert space $\mathfrak{V}$ and set $M(\xi) = \delta(\xi)^{\frac{1}{2}}\Delta(\xi)^{-\frac{1}{2}}L(\xi)$. Let ρ be a representation of $\mathfrak{n}$ on $\mathfrak{V}$ such that $\rho | \mathfrak{h} = dM$ and $\rho(\mathrm{ad}\,\xi X) = M(\xi)\rho(X)M(\xi)^{-1}$ for $\xi \in \Gamma$ and $X \in \mathfrak{n}$. As in [1], form the space $\mathfrak{F}$ of functions from G to $\mathfrak{V}$ determined by Γ and L. This paper, we shall denote this space by $\mathfrak{F}^L$ to show its dependence on L. Let $\mathfrak{F}^{L,\rho}$ be the linear space of all $f \in \mathfrak{F}^L$ such that

(1) the restriction of f to any integral manifold of $\mathfrak{n}^{\square} + \mathfrak{n}^{\square^+}$ is of class C^ω, and

(2) $[X^{\square} + \rho(X)]f = 0$ for all $X \in \mathfrak{n}$.

It is possible that $\mathfrak{F}^{L,\rho} = \{0\}$. In any case, condition (2) is automatically satisfied for $X \in \mathfrak{h}_c$ by any $f \in \mathfrak{F}^L$. It is easy to see that U^L sends $\mathfrak{F}^{L,\rho}$ into itself. The restriction of U^L to $\mathfrak{F}^{L,\rho}$ (or rather to $\mathfrak{H}^{L,\rho}$, the corresponding pre-Hilbert space) is called the *representation of G induced by L and ρ*. It is denoted $U^{L,\rho}$.

In order to show that $\mathfrak{H}^{L,\rho}$ is complete, we shall characterize the $f \in \mathfrak{F}^{L,\rho}$ in terms of our partially complex structure $\mathcal{C}$ on $\mathcal{M}$. By a *charting function* for L and ρ will be meant a C^ω function H with values in $GL(\mathfrak{V})$ such that:

(1) $\mathrm{dom}(H) = \pi^{-1}(O)$, where O is some relatively compact cubical neighborhood in $(\mathcal{M}, \mathcal{C})$ over which is defined a C^ω cross-section in G;

(2) $H(\xi x) = M(\xi)H(x)$ for $\xi \in \Gamma$, $x \in \pi^{-1}(O)$; and

(3) $[X^{\square} + \rho(X)]H = 0$ on $\pi^{-1}(O)$ for all $X \in \mathfrak{n}$.

LEMMA 1. *Let $p_0 \in \mathcal{M}$.* There exists a charting function H such that $p_0 \in \pi(\mathrm{dom}(H))$.

Proof. Let O_1 be a relatively compact cubical neighborhood of p_0 over which a C^ω cross-section α into G is defined. Then the map $\omega: \Gamma \times O_1$ into

G given by $\omega(\xi, p) = \xi\alpha(p)$ is a C^ω diffeomorphism $\Gamma \times O_1$ with $\pi^{-1}(O_1)$. Now equations (3) are C^ω, non-singular, and satisfy the (complex) condition of Frobenius for local integrability. Hence there is a neighborhood N of $\alpha(p_0)$ for which a C^ω function H_1 from N to $GL(\mathfrak{V})$ is defined satisfying (3). Moreover, we may assume that $N = \omega(U \times O)$, where U is a cubical neighborhood of e in Γ and O is a cubical neighborhood of p_0. We define H on $\pi^{-1}(O)$ by $H(\xi\alpha(p)) = M(\xi)H_1(\alpha(p))$ for all $\xi \in \Gamma$ and $p \in O$. Note that H coincides with H_1 on N since (3) implies (2) whenever x and $\xi x \in N$, $\xi \in \Gamma$ (connectivity implies $\xi \in \Gamma_0$). H satisfies (2) by construction and satisfies (3) in N. Now if $X \in \mathfrak{g}$ and $x(.)$ is the 1-parameter subgroup of G lying beneath X, we have

$$\begin{aligned}(X^{\square}H)(\xi g) &= -D_t[H(x(t)\xi g)]\,|_{t=0} \\ &= -D_t[H(\xi\cdot\xi^{-1}x(t)\xi g]\,|_{t=0} = M(\xi)[(\operatorname{ad}\xi^{-1}X)^{\square}H](g),\end{aligned}$$

for $\xi \in \Gamma$, $g \in \pi^{-1}(O)$. Moreover

$$\rho(X)H(\xi g) = M(\xi)M(\xi)^{-1}\rho(X)\,M(\xi)H(g) = M(\xi)\rho(\operatorname{ad}\xi^{-1}X)H(g)$$

for $X \in \mathfrak{n}$. Since $(\operatorname{ad}\Gamma)(\mathfrak{n}) = \mathfrak{n}$, we conclude that H satisfies (3) in $\Gamma N = \pi^{-1}(O)$.

Fix a charting function H with domain $\pi^{-1}(O)$. Let f be a measurable function from $\pi^{-1}(O)$ to $\mathfrak{V}$. f satisfies condition (1) in the definition of $\mathfrak{F}^{L,\rho}$ if and only if $H^{-1}f$ does. Moreover

$$\begin{aligned}[X^{\square} + \rho(X)]f &= [X^{\square} + \rho(X)]\,(HH^{-1}f) \\ &= ([X^{\square} + \rho(X)]H)\,(H^{-1}f) + H[X^{\square}(H^{-1}f)] = H[X^{\square}(H^{-1}f)]\end{aligned}$$

for all $X \in \mathfrak{n}$. Hence f satisfies (2) in the definition of $\mathfrak{F}^{L,\rho}$ if and only if $X^{\square}(H^{-1}f) = 0$ for all $X \in \mathfrak{n}$. Now if $f \in \mathfrak{F}^L$, then $H^{-1}f$ is a measurable function $\pi^{-1}(O)$ to $\mathfrak{V}$ which is constant on the right Γ cosets of G. Thus $H^{-1}f = {}^Hf \circ \pi$, where ${}^Hf$ is a unique measurable function from O to $\mathfrak{V}$. It is easily see that $H^{-1}f$ satisfies condition (1) in the definition of $\mathfrak{F}^{H,\rho}$ if and only if ${}^Hf$ is of class C^ω on the integral manifolds of $\mathfrak{m} + \mathfrak{m}^+$. Moreover $\mathfrak{n}^{\square}(H^{-1}f) = 0$ if and only if $\mathfrak{m}\,{}^Hf = 0$. We have proved

PROPOSITION 2. *Let $\mathfrak{D}$ be a collection of charting functions for L and ρ such that $G = \cup[\operatorname{dom}(H) : H \in \mathfrak{D}]$. Let $f \in \mathfrak{F}^L$. Then $f \in \mathfrak{F}^{L,\rho}$ if and only if ${}^Hf$ is a complex analytic function on the canonical complex submanifolds of $(\mathfrak{M}, \mathfrak{B})$ for all $H \in \mathfrak{D}$.*

Let H be a charting function for L and ρ. Let ϕ be a coordinate map in $\mathfrak{B}$ with domain $\pi(\operatorname{dom}(H))$ and image O. Let α be a cross-section of G

over $\pi(\mathrm{dom}(H))$ and let $\beta = \alpha \circ \phi^{-1}$. Let ω be the homeomorphism of $\Gamma \times O$ onto $\mathrm{dom}(H)$ given by $\omega(\xi, p) = \xi\beta(p)$. Let μ be the measure on $\Gamma \times O$ induced under ω by right Haar measure on G, and let γ be the density of μ with respect to the cartesian product of right Haar measure on Γ by Lebesgue measure on O. We know that $\gamma(\xi, p) = \delta(\xi)^{-1}\Delta(\xi)\gamma(e, p)$ for $\xi \in \Gamma$ and $p \in O[1]$. Finally, for each compact $K \subseteq O$ let

$$R_k = \mathrm{Min}[\gamma(e, p) \| H(\beta(p))^{-1} \|^{-2} : p \in K].$$

Let $f \in \mathfrak{F}^L$ and let K be a compact set in O. Choose $h \in C^+{}_0(\mathrm{dom}(H))$ such that $0 \leqq \tau h \leqq 1$ and $\tau h = 1$ on K. Then

$$\begin{aligned}
\| f \|^2 &\geqq \int_{\mathfrak{M}} \tau h \, d\mu_f = \int_G \| f(x) \|^2 h(x) \, dx \\
&= \int_{\pi^{-1}(K)} \| H(x)^H f(\pi(x)) \|^2 h(x) \, dx \\
&\geqq \int_K \int_\Gamma \| H(\xi\beta(p))^{-1} \|^{-2} \| {}^H f(\phi^{-1}(p)) \|^2 h(\xi\beta(p)) \gamma(\xi, p) \, d\xi \, dp \\
&\geqq R_K \int_K \| {}^H f(\phi^{-1}(p)) \|^2 \, dp
\end{aligned}$$

because $\| H(\xi\beta(p))^{-1} \|^{-2} \gamma(\xi, p)$ is independent of ξ and $\tau h = 1$ on K.

If $\{f_n\} \subseteq \mathfrak{H}^{L,\rho}$ and $f_n \to f \in \mathfrak{H}^L$, we have, for each compact $K \subseteq O$, that ${}^H f_n \circ \phi^{-1}$ and ${}^H f \circ \phi^{-1}$ are in $L_2(K)$ and that ${}^H f_n \circ \phi^{-1} \to {}^H f \circ \phi^{-1}$ in $L_2(K)$. Moreover ${}^H f_n \circ \phi^{-1}$ is complex analytic in its complex parameters, whence ${}^H f$ is complex analytic on the canonical complex submanifolds of $(\mathfrak{M}, \mathfrak{B})$. Applying Lemma 1 and Proposition 2, we obtain

THEOREM 2. *$\mathfrak{H}^{L,\rho}$ is complete and $U^{L,\rho}$ is a unitary representation of G on $\mathfrak{H}^{L,\rho}$.*

4. An intertwining number theorem. Suppose now that with G we are given two collections $\{\Gamma_i, \delta_i, n_i, \mathfrak{V}_i, L_i, \rho_i\}$, $i = 1, 2$, satisfying the conditions of the previous section. We wish to calculate an upper bound on the dimension $I(U^{L_1,\rho_1}, U^{L_2,\rho_2})$ of the space $\mathfrak{R}(U^{L_1,\rho_1}, U^{L_2,\rho_2})$ of operators intertwining U^{L_1,ρ_1} and U^{L_2,ρ_2}. We shall be aided in this by the fact that U^{L_i,ρ_i} is a subrepresentation of U^{L_i}. Define the (non-unitary) representation M_i corresponding to L_i as in Section 3. For $f \in C_0{}^\infty(G)$ and $v \in \mathfrak{V}_i$, set

$$\epsilon_i{}^0(f, v)(x) = \int_{\Gamma_i} f(\xi x) M_i(\xi)^{-1} v \, d\xi.$$

Let P_i be the projection of H^{L_i} ond H^{L_i,ρ_i} and define $\epsilon_i(f,v) = P_i\epsilon_i^0(f,v)$. If $A \in \mathcal{R}(U^{L_1}, U^{L_2})$, $f \in C_0^\infty(G)$, and $v \in \mathfrak{V}_1$, set $r^0{}_A(f)v = (A\epsilon_1{}^0(f,v))(e)$. This definition makes sense: see [1], § 5. If $A \in \mathcal{R}(U^{L_1,\rho_1}, U^{L_2,\rho_2})$, then $AP_1 = P_2AP_1 \in \mathcal{R}(U^{L_1}, U^{L_2})$ so that we may set $r_A(f)v = r^0{}_{AP_1}(f)v$. If $x, y \in G$, we set $(\zeta_{x,y}f)(z) = f(x^{-1}zy)$ for $f \in C_0^\infty(G)$.

If S is an operator mapping $C_0^\infty(G)$ into itself, the operator ${}^tS$, if it exists, will be defined by the identity $\int (Sf)(x)g(x)\,dx = \int f(x)({}^tSg)(x)\,dx$ for all $f, g \in C_0^\infty(G)$. It is easy to check that if $X \in \mathfrak{g}_c$ and $x, y \in G$, then ${}^tX = -X$, ${}^t(X^\square) = -X^\square + (X\Delta)(e)$, and ${}^t\zeta_{x,y} = \Delta(x)\zeta_{x^{-1},y^{-1}}$. If Z is any vector-valued distribution (in the sense of Schwartz [6]) on G, we set $SZ = Z \circ {}^tS$. Finally, any locally integrable function F on G will be identified with the distribution $f \to \int f(x)F(x)\,dx$.

Our intertwining number theorem now goes as follows:

THEOREM 3. *Let X_0 be an element of the universal associative enveloping algebra of $\mathfrak{g}$ which, as a left invariant differential operator on G, is elliptic of order $> \frac{1}{2}\dim(G/\Gamma_2)$. For $f \in C_0^\infty(G)$, set $\|f\|_{X_0} = \|X_0f\|_\infty + \|f\|_\infty$. For each relatively compact open set O of G, give $C_0^\infty(O)$ the topology induced by $\|\,.\,\|_{X_0}$; give $C_0^\infty(G)$ the inductive limit topology. Let $\mathfrak{M}$ be the subspace of maps $Z \in \mathcal{L}(C_0^\infty(G);\ \mathcal{L}(\mathfrak{V}_1, \mathfrak{V}_2))$ which satisfy the distribution equations:*

(1) $\zeta_{\xi_1,\xi_2}Z = M_2(\xi_2)^{-1}ZM_1(\xi_1)^{-1*}$ *for* $\xi_i \in \Gamma_i$;

(2) $X^{\square+}Z = Z\rho_1(X)^*$, $X \in \mathfrak{n}_1$;

(3) $XZ = -\rho_2(X)Z$, $X \in \mathfrak{n}_2$.

Then $I(U^{L_1,\rho_1}, U^{L_2,\rho_2}) \leqq \dim \mathfrak{M}$.

Proof. From [1], Theorem 3, we deduce that $A \to r_A$ is a faithful linear map of $\mathcal{R}(U^{L_1,\rho_1}, U^{L_2,\rho_2})$ into the subspace of maps $Z \in \mathcal{L}(C_0^\infty(G); \mathcal{L}(\mathfrak{V}_1; \mathfrak{V}_2))$ satisfying (1). To show that r_A satisfies (2), we note that

$$(\epsilon_1(f,v), g) = \int_G f(x)(v, g(x))\,dx$$

for $f \in C_0^\infty(G)$, $v \in \mathfrak{V}_1$, and $g \in \mathfrak{F}^{L_1,\rho_1}$. This follows immediately from the definition of ϵ_1 and the proof of Lemma 2(a) in [1]. Now $(v, g(.))$, regarded as a distribution on G, satisfies the equation $X^{\square+}(v, g(.)) = (\rho_1(X)^*v, g(.))$ for $X \in \mathfrak{n}_1$, in virtue of properties (1) and (2) of $\mathfrak{F}^{L_1,\rho_1}$. Therefore,

$$\begin{aligned}(\epsilon_1({}^tX^{\square+}f, v), g) &= \int ({}^tX^{\square+}f)(x)(v, g(x))\,dx \\ &= \int f(x)(\rho_1(X)^*v, g(x))\,dx = (\epsilon_1(f, \rho_1(X)^*v), g).\end{aligned}$$

over $\pi(\mathrm{dom}(H))$ and let $\beta = \alpha \circ \phi^{-1}$. Let ω be the homeomorphism of $\Gamma \times O$ onto $\mathrm{dom}(H)$ given by $\omega(\xi, p) = \xi\beta(p)$. Let μ be the measure on $\Gamma \times O$ induced under ω by right Haar measure on G, and let γ be the density of μ with respect to the cartesian product of right Haar measure on Γ by Lebesgue measure on O. We know that $\gamma(\xi, p) = \delta(\xi)^{-1}\Delta(\xi)\gamma(e, p)$ for $\xi \in \Gamma$ and $p \in O[1]$. Finally, for each compact $K \subseteq O$ let

$$R_k = \mathrm{Min}[\gamma(e, p) \| H(\beta(p))^{-1} \|^{-2} : p \in K].$$

Let $f \in \mathfrak{F}^L$ and let K be a compact set in O. Choose $h \in C^+{}_0(\mathrm{dom}(H))$ such that $0 \leqq \tau h \leqq 1$ and $\tau h = 1$ on K. Then

$$\begin{aligned} \| f \|^2 &\geqq \int_{\mathfrak{M}} \tau h \, d\mu_f = \int_G \| f(x) \|^2 h(x) \, dx \\ &= \int_{\pi^{-1}(K)} \| H(x)^H f(\pi(x)) \|^2 h(x) \, dx \\ &\geqq \int_K \int_\Gamma \| H(\xi\beta(p))^{-1} \|^{-2} \| {}^H f(\phi^{-1}(p)) \|^2 h(\xi\beta(p)) \gamma(\xi, p) \, d\xi \, dp \\ &\geqq R_K \int_K \| {}^H f(\phi^{-1}(p)) \|^2 \, dp \end{aligned}$$

because $\| H(\xi\beta(p))^{-1} \|^{-2} \gamma(\xi, p)$ is independent of ξ and $\tau h = 1$ on K.

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Then $I(U^{L_1,\rho_1}, U^{L_2,\rho_2}) \leqq \dim \mathfrak{M}$.

Proof. From [1], Theorem 3, we deduce that $A \to r_A$ is a faithful linear map of $\mathcal{R}(U^{L_1,\rho_1}, U^{L_2,\rho_2})$ into the subspace of maps $Z \in \mathcal{L}(C_0^\infty(G); \mathcal{L}(\mathfrak{V}_1; \mathfrak{V}_2))$ satisfying (1). To show that r_A satisfies (2), we note that

$$(\epsilon_1(f,v), g) = \int_G f(x)(v, g(x))\,dx$$

for $f \in C_0^\infty(G)$, $v \in \mathfrak{V}_1$, and $g \in \mathfrak{F}^{L_1,\rho_1}$. This follows immediately from the definition of ϵ_1 and the proof of Lemma 2(a) in [1]. Now $(v, g(.))$, regarded as a distribution on G, satisfies the equation $X^{\square+}(v, g(.)) = (\rho_1(X)^*v, g(.))$ for $X \in \mathfrak{n}_1$, in virtue of properties (1) and (2) of $\mathfrak{F}^{L_1,\rho_1}$. Therefore,

$$\begin{aligned}(\epsilon_1({}^tX^{\square+}f, v), g) &= \int ({}^tX^{\square+}f)(x)(v, g(x))\,dx \\ &= \int f(x)(\rho_1(X)^*v, g(x))\,dx = (\epsilon_1(f, \rho_1(X)^*v), g).\end{aligned}$$

From this we obtain

$$(X^{\square^+} r_A)(f)v = (A\epsilon_1({}^tX^{\square^+} f, v))(e)$$
$$= (A\epsilon_1(f, \rho_1(X)^* v))(e) = r_A(f)\rho_1(X)^* v$$

for $X \in \mathfrak{n}_1$.

As for property (3), we recall from the proof of Theorem 3 in [1] that $r_A(\zeta_{e,y} f)v = (A\epsilon_1(f, v))(y)$ for all $y \in G$. Now the mapping $y \to \zeta_{e,y} f$ is a C^∞ map of G into $C_0^\infty(G)$. Since r_A is continuous and linear, we see that $(A\epsilon_1(f, v))(.)$ is a C^∞ function and that $r_A(Xf)v = [X(A\epsilon_1(f, v))](e)$ $= -[X^{\square}(A\epsilon_1(f, v))](e)$ for all $X \in \mathfrak{g}_c$. If $X \in \mathfrak{n}_2$, we conclude that $r_A(Xf)v = \rho_2(X) r_A(f)v$.

5. Totally complex subalgebras. A complex subalgebra $\mathfrak{n}$ of $\mathfrak{g}_c$ will be called *totally complex* if $\mathfrak{n} + \mathfrak{n}^+ = \mathfrak{g}_c$.

PROPOSITION 3. *In Theorem* 3, *let either* $\mathfrak{n}_1$ *or* $\mathfrak{n}_2$ *be totally complex. Then* $\mathfrak{M}$ *consists entirely of* C^ω *functions.*

Proof. We shall give a proof for the case when $\mathfrak{n}_2$ is totally complex. The other case is entirely analogous. Let $\{X_1, \cdots, X_n\}$ be a basis for $\mathfrak{h}_2$, and extend it to a basis $\{X_1, \cdots, X_n, X_{n+1}, \cdots, X_r\}$ of $\mathfrak{n}_2$. Then $\{X_1, \cdots, X_r, X_{n+1}^+, \cdots, X_r^+\}$ is a basis of $\mathfrak{g}_c$, by our assumption. If we write $X_j = Y_j + iZ_j$, where $Y_j, Z_j \in \mathfrak{g}$ for $j = \mathfrak{n}+1, \cdots, r$, it follows that $\{X_1, \cdots, X_n, Y_{n+1}, \cdots, Y_r, Z_{n+1}, \cdots, Z_r\}$ is a basis of $\mathfrak{g}$. Now the differential operator $S = \sum_j {}^{r_1} X_j^+ (X_j + \rho_2(X_j)) = -\sum_j {}^{n_1} X_j^2 - \sum_j {}^{r_{n+1}} Y_j^2 - \sum_j {}^{r_{n+1}} Z_j^2 +$ terms of lower order. This shows that S is elliptic (cf. [7]) and C^ω. If $Z \in \mathfrak{M}$, then $SZ = 0$ according to property (3) of $\mathfrak{M}$. Therefore, Z is a C^ω function (see p. 146 of [6] and the references given there).

Let $\mathfrak{n}$ be a totally complex subalgebra of $\mathfrak{g}_c$. We shall say that x is $\mathfrak{n}$-*regular* if $\mathfrak{n} + \operatorname{ad} x^{-1}\mathfrak{n}^+ = \mathfrak{g}_c$ and shall denote the set of $\mathfrak{n}$-regular elements by $R_{\mathfrak{n}}$.

LEMMA 2. *In Theorem* 3 *assume the following:* G *is connected;* $\Gamma_1 = \Gamma_2 = \Gamma$, *say; and* $\mathfrak{n}_1 = \mathfrak{n}_2 = \mathfrak{n}$, *say, a totally complex subalgebra of* $\mathfrak{g}_c$. *Let* $Z \in \mathfrak{M}$ *and suppose* $Z(x_0) = 0$ *for some* $x_0 \in R_{\mathfrak{n}}$. *Then* $Z = 0$.

Proof. We choose vector fields $X_1, \cdots, X_n \in \mathfrak{n}$ such that, for suitably chosen r, $\{X_1, \cdots, X_r, \operatorname{ad} x_0^{-1} X_{r+1}^+, \cdots, \operatorname{ad} x_0^{-1} X_n^+\}$ is a basis for $\mathfrak{g}_c$. It is then easy to see that there is a C^ω cubical neighborhood N about x_0 such that $\{X_1, \cdots, X_r, \operatorname{ad} x^{-1} X_{r+1}^+, \cdots, \operatorname{ad} x^{-1} X_n^+\}$ is a basis for $\mathfrak{g}_c$ for all $x \in N$. Let

$x_1, \cdots, x_n$ be the coordinate functions for N. There exist C^ω functions a_{ij} for $i, j = 1, \cdots, n$ such that, for all $x \in N$,

$$\left(\frac{\partial}{\partial x_i}\right)_x = \sum_j{}^{r_1} a_{ij}(x)\,(X_j)_x + \sum_j{}^{n}{}_{r+1}\, a_{ij}(x)\,(\mathrm{ad}\, x^{-1}X_j^+)_x.$$

Now for any $X \in \mathfrak{g}_c$, $(\mathrm{ad}\, x^{-1}X)_x = -X^\square{}_x$. Therefore

$$\frac{\partial Z}{\partial x_i} = -\sum_j{}^{r_1} a_{ij}(\,.\,)\rho_2(X_j)Z - \sum_j{}^{n}{}_{r+1}\, a_{ij}(\,.\,)Z\rho_1(X_j)^*$$

by virtue of Theorem 3. But $Z(x_0) = 0$. From the uniqueness theorem for ordinary linear differential equations we conclude that Z vanishes on N. Since Z is a C^ω function, Z vanishes identically on G.

We are now in a position to prove

THEOREM 4. *Suppose that G is connected and suppose the induced representations U^{L_1,ρ_1} and U^{L_2,ρ_2} satisfy the following conditions: L_1 and L_2 are defined on Γ; ρ_1 and ρ_2 are defined on $\mathfrak{n}$, a totally complex subalgebra of $\mathfrak{g}_c$. For each $x \in G$, define a representation ρ_{1x} of $\mathrm{ad}\, x^{-1}\mathfrak{n}^+$ by setting $\rho_{1x}(X) = \rho_1(\mathrm{ad}\, xX^+)^*$ for $X \in \mathrm{ad}\, x^{-1}\mathfrak{n}^+$, also a representation M_{1x} of $x^{-1}\Gamma x$ by setting $M_{1x}(\xi) = M_1(x\xi x^{-1})^{*-1}$ for $\xi \in x^{-1}\Gamma x$. Define $I(x)$ to be the dimension of $\mathcal{R}_x$, where*

$$\mathcal{R}_x = \mathcal{R}\,(\rho_{1x}|\mathfrak{n} \cap \mathrm{ad}\, x^{-1}\mathfrak{n}^+, \rho_2|\mathfrak{n} \cap \mathrm{ad}\, x^{-1}\mathfrak{n}) \cap \mathcal{R}\,(M_{1x}|\Gamma \cap x^{-1}\Gamma x, M_2|\Gamma \cap x^{-1}\Gamma x).$$

Then (1) $I(x)$ depends only on the Γ: Γ double coset to which x belongs and (2) $I(U^{L_1,\rho_1}, U^{L_2,\rho_2}) \leqq \mathrm{GLB}[I(x) : x \in R_\mathfrak{n}] < \infty$.

Proof. First note that $e \in R_\mathfrak{n}$ so that $\mathrm{GLB}[I(x) : x \in R_\mathfrak{n}] < \infty$. Fix a point $x_0 \in R_\mathfrak{n}$. For each $Z \in M$, set $\Omega(T) = Z(x_0)$. The map $\Omega \in \mathcal{L}(\mathfrak{M}; \mathcal{L}(\mathfrak{V}_1; \mathfrak{V}_2))$.

(a) $\Omega \in \mathcal{L}(\mathfrak{M}; \mathcal{R}_{x_0})$. In fact, let $Z \in \mathfrak{M}$. If $\xi \in \Gamma \cap x_0^{-1}\Gamma x_0$, then $x_0\xi x_0^{-1} \in \Gamma$ and $(x_0\xi x_0^{-1})x_0\xi^{-1} = x_0$. Property (1) of $\mathfrak{M}$ implies that $Z(x_0) = M_2(\xi)Z(x_0)M_1(x_0\xi x_0^{-1})^*$ so that $Z(x_0)M_{1x_0}(\xi) = M_2(\xi)Z(x_0)$. Again, if $X \in \mathfrak{n} \cap \mathrm{ad}\, x_0^{-1}\mathfrak{n}^+$, then $\mathrm{ad}\, x_0X^+ \in \mathfrak{n}$. Properties (2) and (3) of $\mathfrak{M}$ imply that

$$\begin{aligned} Z(x_0)\rho_{1x_0}(X) &= Z(x_0)\rho_1(\mathrm{ad}\, x_0X^+)^* \\ &= ([\mathrm{ad}\, x_0X]^\square Z)(x_0) = -(XZ)(x_0) = \rho_2(X)Z(x_0). \end{aligned}$$

Therefore $\Omega(Z) \in \mathcal{R}_{x_0}$.

(b) Ω is one-to-one. In fact, suppose $\Omega(Z) = 0$. Then $Z(x_0) = 0$ so that $Z = 0$ by Lemma 3.

We have shown that Ω is an isomorphism of $\mathfrak{M}$ into $\mathcal{R}_{x_0}$. This proves statement (2). The proof of statement (1) is standard and will be omitted.

COROLLARY 1. *In case* π *is totally complex, a necessary condition for* $\mathfrak{H}^{L,\rho} \neq \{0\}$ *is that* $\delta = \Delta|\Gamma$.

Proof. In Theorem 4, let $\mathfrak{B}_1 = \mathfrak{B}_2 = \mathfrak{B}$, $L_1 = L_2 = L$, and $\rho_1 = \rho_2 = \rho$. If $\mathfrak{H}^{L,\rho} \neq \{0\}$, then $\mathfrak{M} \neq \{0\}$ so that $\mathcal{R}_e \neq \{0\}$. Let $0 \neq S \in \mathcal{R}_e$. Then, for all $\xi \in \Gamma$, $\delta(\xi)^{-\frac{1}{2}}\Delta(\xi)^{\frac{1}{2}} SL(\xi) = \delta(\xi)^{\frac{1}{2}}\Delta(\xi)^{-\frac{1}{2}} L(\xi)S$. It follows that

$$\delta(\xi)\Delta(\xi)^{-1} \| Sv \| = \| SL(\xi)v \| \leqq \| S \| \| v \|$$

for all $v \in \mathfrak{B}$. But this implies that $\delta(\xi)\Delta(\xi)^{-1} \| S \| \leqq \| S \|$ for all $\xi \in \Gamma$, and our conclusion follows.

COROLLARY 2. *Suppose* π *is totally complex. If* L *is irreducible and if* $\mathfrak{H}^{L,\rho} \neq \{0\}$, *then* $U^{L,\rho}$ *is irreducible. If* L' *and* ρ' *are two more such representations defined on* Γ *and* π, *then* $U^{L',\rho'}$ *and* $U^{L,\rho}$ *are inequivalent if* L' *and* L *are inequivalent.*

Proof. Apply Theorem 4 to (L,ρ) *and* (L',ρ'). We get $I(U^{L,\rho}, U^{L',\rho'}) \leqq I(e)$, which equals 1 or 0 according as L and L' are equivalent or inequivalent. Our corollary follows immediately.

7. Some examples.

(a) Some representations of Harish-Chandra.

Let G be a real semi-simple Lie group and let Γ be a Cartan subgroup. In this case, $\Delta = 1$ and $\delta = 1$. We suppose that the image of Γ in the adjoint group of G is compact. Let $\mathfrak{g}$ (resp. $\mathfrak{h}$) be the Lie algebra of G (resp. Γ). $\mathfrak{h}_c$ is a Cartan subalgebra of $\mathfrak{g}_c$, and our assumption implies that every root of $\mathfrak{g}_c$ with respect to $\mathfrak{h}_c$ is purely imaginary on $\mathfrak{h}$. Let the roots be totally ordered in the usual way, and set $\mathfrak{n}$ equal to $\mathfrak{h}_c$ plus the space spanned by the root vectors belonging to the positive roots. Let X be a root vector belonging to the positive root α. Let $H \in \mathfrak{h}$. Then $[H, X^+] = -[H^+, X]^+ = [H, X]^+ = (\alpha(H)X)^+ = -\alpha(H)X^+$ since $\alpha(H)$ is pure imaginary. We see that $\mathfrak{n}^+$ is $\mathfrak{h}_c$ plus the space spanned by the root vectors belonging to the negative roots. We have that $\mathfrak{n}$ is a complex subalgebra of $\mathfrak{g}_c$, that $\mathfrak{n} + \mathfrak{n}^+ = \mathfrak{g}_c$, and that $\mathfrak{n} \cap \mathfrak{n}^+ = \mathfrak{h}_c$. Since Γ is connected, $(\mathrm{ad}\,\Gamma)\mathfrak{n} = \mathfrak{n}$.

Let χ be a character of the abelian group Γ. $d\chi$ extends uniquely to a complex linear representation ρ of $\mathfrak{n}$ on $\boldsymbol{C}$: merely note that $\rho(X)$ must vanish whenever X is a root vector belonging to a positive root. *Let* $\mathbf{X}$ *be the*

collection of all characters χ *of* Γ *such that* $\mathfrak{H}^{\chi,\rho} \neq \{0\}$. *Then the representations* $U^{\chi,\rho}$ *for* $\chi \in \mathbf{X}$ *are all irreducible and mutually inequivalent.* The actual structure of $\mathbf{X}$ has been studied by Harish-Chandra in [4].

(b) Certain representations of $GL(2k, \boldsymbol{R})$.

Let $\mathfrak{W}$ be a real finite dimensional vector space and let $\mathfrak{g}$ be the Lie algebra of all $T \in \mathfrak{L}(\mathfrak{W}; \mathfrak{W})$ with vanishing trace. $\mathfrak{g}_c$ may then be identified with the Lie algebra of all $T \in \mathfrak{L}(\mathfrak{W}_c; \mathfrak{W}_c)$ with vanishing trace. Suppose $\dim \mathfrak{W} = 2k$, k integral. It is then possible to choose vectors $v_1, \cdots, v_k \in \mathfrak{W}_c$ such that $\{v_1, \cdots, v_k, v_1^+, \cdots, v_k^+\}$ is a basis for $\mathfrak{W}_c$. Let $\mathfrak{n} = \{T \in \mathfrak{g}_c : v_j$ is an eigenvector of T for $j = 1, \cdots, k\}$, a Lie subalgebra $\mathfrak{g}_c$. Then $\mathfrak{h}_1 = \mathfrak{n} \cap \mathfrak{n}^+$ equals $\{T \in \mathfrak{g}_c : v_j$ and v_j^+ is an eigenvector of T for $j = 1, \cdots, k\}$, a Cartan subalgebra of $\mathfrak{g}_c$, and $\mathfrak{h} = \mathfrak{h}_1 \cap \mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{h}_c = \mathfrak{h}_1$.

We define linear functionals $\lambda_1, \cdots, \lambda_k$ on $\mathfrak{n}$ by $Tv_j = \lambda_j(T)v_j$ for $T \in \mathfrak{n}$. It is easily seen that the derived algebra $\mathfrak{n}^{(2)}$ of $\mathfrak{n} = \{T \in \mathfrak{n} : \lambda_j(T) = 0$ for $j = 1, \cdots, k\}$. Our first object is to determine the space $\mathbf{P}$ of all homomorphisms of $\mathfrak{n}$ into $\boldsymbol{C}$ such that $\rho(T)$ *is* pure imaginary for $\rho \in \mathbf{P}$, $T \in \mathfrak{h}$.

LEMMA 3. $\mathbf{P} = \{a \sum_j k_j \lambda_j : a \in \boldsymbol{R}\}$.

Proof. Clearly $\mathbf{P}$ is the set of all linear functionals ρ on $\mathfrak{n}$ such that $\rho(\mathfrak{n}^{(2)}) = 0$ and $\rho(T^+) = \overline{\rho(T)}$ for $T \in \mathfrak{h}_c$. We first show that $\sum_j k_j \lambda_j \in \mathbf{P}$. Clearly $\sum_j k_j \lambda_j(\mathfrak{n}^{(2)}) = 0$. Let $T \in \mathfrak{h}_c$. Then $T^+ v_j^+ = -(Tv_j)^+ = -\overline{\lambda_j(T)} v_j^+$. Since T has vanishing trace, we conclude that $\sum_j k_j \lambda_j(T^+) = \sum_j k_j \overline{\lambda_j(T)}$.

Conversely, let $\rho \in \mathbf{P}_0$. Suppose $T \in \mathfrak{n}$ satisfies $\sum_j k_j \lambda_j(T) = 0$. Define $S \in \mathfrak{L}(\mathfrak{W}_c; \mathfrak{W}_c)$ by $Sv_j = \lambda_j(T)v_j$, $Sv_j^+ = 0$ for $j = 1, \cdots, k$. By assumption, $S \in \mathfrak{g}_c$ so that $S \in \mathfrak{h}_c$. It is also clear that $T - S$ and $S^+ \in \mathfrak{n}^{(2)}$. Therefore $\rho(T) = \rho(T - S) + \overline{\rho(S^+)} = 0$. We conclude that $\rho = a \sum_j k_j \lambda_j$ for some $a \in \boldsymbol{C}$. If $T \in \mathfrak{h}_c$, then $\rho(T) = \overline{\rho(T^+)} = \bar{a} \sum_j k_j \lambda_j(T)$. Since $\sum_j k_j \lambda_j$ does not vanish on $\mathfrak{h}_c$, we see that $a \in \boldsymbol{R}$.

Let $G = SL(\mathfrak{W})$ and let Γ be the analytic subgroup of G lying beneath $\mathfrak{h}$. Let $\mathbf{P}_0$ be the collection of all $\rho \in \mathbf{P}$ such that $\rho | \mathfrak{h} = d\chi$ for some (necessarily unique) character χ of Γ. $\mathfrak{n} = \mathfrak{h} + \mathfrak{n}^{(2)}$ implies that χ uniquely determines ρ. A necessary and sufficient condition that $\rho \in \mathbf{P}_0$ is that $\rho \in \mathbf{P}$ and $i\rho$ be integral on the kernel of the map $T \to \exp T$ of $\mathfrak{h}$ onto Γ, and this is easily seen to be equivalent to the requirement that $i\rho(T)$ be integral for all

$T \in \mathfrak{h}$ such that $\operatorname{Im} \lambda_j(T)$ is integral for $j = 1, \cdots, k$. But $T \to (\operatorname{Im} \lambda_1(T), \cdots, \operatorname{Im} \lambda_k(T))$ maps $\mathfrak{h}$ *onto* $\boldsymbol{R}^k$ because the trace restriction on $T \in \mathfrak{h}$ is just $\sum_j^{k_1} \operatorname{Re} \lambda_j(T) = 0$. We conclude that $\rho \in \mathbf{P}_0$ if and only if $\rho = m \sum_j^{k_1} \lambda_j$, where m is integral.

Since both G and Γ are unimodular and since Γ is connected, we obtain a series of representations $U^{\chi,\rho}$ of G, where $\rho \in \mathbf{P}_0$ and $d\chi = \rho | \mathfrak{h}$. If $\mathbf{P}_{00}$ is the set of $\rho \in \mathbf{P}_0$ for which $\mathfrak{H}^{\chi,\rho} \neq \{0\}$, we conclude from Corollary 2 of Theorem 4: *the representations* $U^{\chi,\rho}$ *for* $\rho \in \mathbf{P}_{00}$ *are all irreducible and mutually inequivalent.* These representations are mentioned by Gelfand and Graev in Subsection 4 of the Introduction of [3].

THE UNIVERSITY OF CALIFORNIA,
LOS ANGELES, CALIFORNIA.

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ON SOME INTERPOLATION PROBLEMS FOR ANALYTIC FUNCTIONS.*

By H. S. Shapiro and A. L. Shields.[1]

In the first part of this paper we consider an interpolation problem for bounded analytic functions that was first solved by Lennart Carleson [3] (a detailed description is given below). We give a new proof of Carleson's result and we obtain analogous results for the spaces H_p. In the second part of the paper (Sections Four and Five) we study a related interpolation problem in certain Hilbert spaces of analytic functions, using the results obtained previously.

The interpolation problem for bounded functions has to do with the existence of sequences of points $\{z_n\}$ in the unit circle having the property that, given an arbitrary bounded sequence of complex numbers $\{w_n\}$, there is a bounded analytic function f for which

$$f(z_n) = w_n \qquad (n = 1, 2, \cdots).$$

If the sequence $\{z_n\}$ has this property we shall call it an *interpolating sequence.*

The problem of the existence of interpolating sequences was posed by R. C. Buck. The classical interpolation theory (see, for example, Nevanlinna [10] and Schur [15]) does not seem to be of help here. For example, Schur gives an explicit algorithm for finding the bounded function of minimal norm taking prescribed values $w_1, \cdots, w_n$ at the points $z_1, \cdots, z_n$. However, his method does not seem to lead to an effective estimate of this norm as n increases and as $\{w_n\}$ ranges over the unit sphere of l_∞.

Independently of one another L. Carleson [3], W. Hayman [6] and D. J. Newman [11] studied this problem. Carleson showed that a necessary and sufficient condition for $\{z_n\}$ to be an interpolating sequence is that there exist a positive number δ such that

$$\text{(C)} \qquad \prod_{n \neq k} |(z_k - z_n)/(1 - \bar{z}_n z_k)| \geqq \delta \qquad (k = 1, 2, \cdots).$$

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Newman proved that condition (C) together with condition (H) given below are necessary and sufficient fo r$\{z_n\}$ to be an interpolating sequence:

$$\text{(H)} \qquad \sum |f(z_n)|(1-|z_n|) < \infty \qquad \text{(for all } f \in H_1\text{)}.$$

Thus Carleson's result tells us that (C) implies (H), and in fact this is the heart of his proof.

Hayman showed that (C) is necessary and that a slight strengthening of it is sufficient. With his stronger condition he is able to give an explicit interpolation formula to find an f taking the given values $\{w_n\}$. He also shows that if one can interpolate arbitrary sequences of zeros and ones at the points $\{z_n\}$ then condition (C) is satisfied.

Both Newman and Hayman show that if z_n tend exponentially to the boundary, that is, if $\limsup(1-|z_{n-1}|)/(1-|z_n|) < 1$, then $\{z_n\}$ is an interpolating seqence. They point out that if the $\{z_n\}$ all lie on one radius, then this condition is also necessary.

Three years earlier, in connection with an interpolation problem for functions of bounded characteristic, A. G. Naftalevitch [9] showed that if $\{z_n\}$ satisfy the condition $\sum(1-|z_n|) < \infty$, then there exist points $\{z_n'\}$, with $|z_n'| = |z_n|$ $(n = 1, 2, \cdots)$, for which condition (C) is satisfied and for which

$$\text{(N)} \qquad \sum(1-|z_n|)\log^+|f(z_n)| < \infty$$

for all analytic f of bounded characteristic. He did not show that condition (N) is implied by condition (C).

Subsequently, V. Kabaila [7] pointed out that Naftalevitch's method shows that if $\sum(1-|z_n|) < \infty$, then there are points $\{z_n'\}$, with $|z_n'| = 1 z_n|$ $(n = 1, 2, \cdots)$, for which condition (H) is satisfied.

In Section Two of this paper we give a new proof of Carleson's result that $\{z_n\}$ is an interpolating sequence if and only if condition (C) is satisfied. We first show by a Banach space argument that, in the presence of condition (C), $\{z_n\}$ is an interpolating sequence if and only if condition (H) is satisfied. This part of the argument is similar to Carleson's proof, though differing in detail. We next show that (H) is equivalent to the following weighted interpolation problem in the space H_2:

For each sequence $\{c_n\} \in l_2$ there is an $f \in H_2$ for which

$$f(z_n)(1-|z_n|^2)^{\frac{1}{2}} = c_n \qquad (n = 1, 2, \cdots).$$

We complete the proof by showing that if condition (C) is satisfied then this weighted interpolation problem can be solved.

In Section Three we discuss weighted interpolation in the spaces H_p. We associate with each $f \in H_p$ the sequence of weighted values

$$T_p f = (f(z_1)(1-|z_1|)^{1/p}, f(z_2)(1-|z_2|)^{1/p}, \cdots),$$

and we consider the following two problems (as always, l_p denotes the space of sequences $\{c_n\}$ for which $\sum |c_n|^p < \infty$).

1. Is $l_p \subset T_p H_p$, that is, can one obtain arbitrary p-th power summable values by weighted interpolation using functions of class H_p?

2. Is $T_p H_p \subset l_p$, that is, do we have $\sum |f(z_n)|^p (1-|z_n|) < \infty$ for all $f \in H_p$?

One can show that if (1) is satisfied for some value of p $(1 \leqq p \leqq \infty)$, then condition (C) is satisfied. A Banach space argument shows that, in the presence of condition (C), (1) holds for a given value of p if and only if (2) holds for the conjugate index q $(1/p + 1/q = 1)$. The problem can then be reduced to the weighted interpolation problem in H_2 which was solved in Section Two. The final result is that for any given value of p $(1 \leqq p \leqq \infty)$ $T_p H_p = l_p$ if and only if the sequence $\{z_n\}$ satisfies condition (C).

In Section Four we consider for general Hilbert spaces of analytic functions a weighted interpolation problem analogous to the H_2 problem in Section Two. Certain equivalences are pointed out and an application is made to the H_2 interpolation problem in case the points z_n approach the boundary exponentially.

In Section Five we use the results on H_2 interpolation and the results of Section Four to study interpolation in two special Hilbert spaces: the space H_2' of functions that are derivatives of H_2 functions, and the space A_2 of functions with finite area norm: $\int\int |f(z)|^2 \, dxdy < \infty$. We show that if the points $\{z_n\}$ satisfy condition (C), then the weighted interpolation problem can be solved in each of these spaces.

Furthermore, a necessary condition for weighted interpolation in these spaces (and in H_2) is that the points $\{z_n\}$ be "weakly separated," that is, $|z_i - z_j| \geqq c(1-|z_i|)$ for some constant $c > 0$ and all i, j. If the $\{z_n\}$ all lie on one radius then this condition is equivalent to exponential approach to the boundary and is both necessary and sufficient. This is related to some results of Epstein, Greenstein, and Minker [4].

1. Background material. As usual H_p $(1 \leqq p < \infty)$ denotes the Banach space of functions f, analytic for $|z| < 1$, for which

$$\|f\|_p = \sup_r \{\tfrac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\}^{1/p} < \infty.$$

H_∞ will denote the bounded analytic functions with the supremum norm. For the properties of the spaces H_p see the book of Privaloff ([12], Chapters 1, 2). In particular we shall need the fact that H_p is isometric to a closed subspace of $L_p(0, 2\pi)$ $(1 \leqq p \leqq \infty)$. If $f \in L_p$, $g \in L_q$ $(1/p + 1/q = 1)$, then we write

$$(f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) g(e^{i\theta}) d(e^{i\theta}). \tag{1}$$

With this notation H_q is the annihilator of H_p: $H_q = H_p{}^\perp$ $(1 \leqq p < \infty)$. In the Hilbert space L_2 (1) is not an inner product.

Duality. Let E be a Banach space with conjugate space E^*. Let $F \subset E$ be a closed subspace and let $F^\perp$ be the set of all those elements of E^* that vanish on F. Then F^*, the conjugate space of F, can be identified with the quotient space $E^*/F^\perp$. The norm in the quotient space is given by

$$\| e^* + F^\perp \| = \inf \| e^* + \lambda \| \qquad (\lambda \in F^\perp).$$

On the other hand, the norm of a linear functional is the supremum of its values on the unit sphere, and so:

$$\inf_{\lambda \in F^\perp} \| e^* + \lambda \| = \sup_{\substack{f \in F \\ \|f\| = 1}} |(f, e^*)|. \tag{2}$$

It follows from the Hahn-Banach theorem that the infimum on the left is always attained.

Similarly, by considering E/F and its conjugate space $F^\perp$ we have:

$$\inf_{f \in F} \| e + f \| = \sup_{\substack{\lambda \in F^\perp \\ \|\lambda\| = 1}} |(e, \lambda)|. \tag{3}$$

The supremum on the right is always attained (since, in the weak star topology, the unit ball of $F^\perp$ is compact and e is continuous). See Havinson [5] and Rogosinski, Shapiro [13] for a discussion of these duality relations and their applications to extremal problems in H_p.

Quadratic forms. Let (a_{ij}) $(i, j = 1, 2, \cdots)$ be given. A result of Schur [14] states that if $\sum_i |a_{ij}| \leqq M$ for all j, and if $\sum_j |a_{ij}| \leqq N$ for all i, then

$$\left| \sum_{i,j} a_{ij} x_i \bar{x}_j \right| \leqq (MN)^{\frac{1}{2}} \sum |x_i|^2. \tag{4}$$

2. Interpolation by bounded analytic functions. Let $\{z_n\}$ be a sequence of distinct points in the unit circle. Define a linear operator T from H_∞ to l_∞ (the bounded sequences) by

$$Tf = (f(z_1), f(z_2), \cdots).$$

Clearly $\| T \| = 1$. Note that if T is onto, then the interpolation can actually be done in a uniformly bounded manner, that is, there is a number M such that if $w = \{w_k\}$ is any element in the unit ball of l_∞, then there is an $f \in H_\infty$ with $Tf = w$ and $\| f \| \leqq M$.

Indeed, let N be the set of those $f \in H_\infty$ vanishing on $\{z_n\}$. Then N is a closed subspace and T induces a one-to-one continuous mapping from the quotient Banach space H_∞/N onto l_∞. By a theorem of Banach ([1], Chap. IV, Théorème 5) the inverse mapping is bounded, which is our assertion.

THEOREM 1. *A necesssary and sufficient condition for $\{z_n\}$ to be an interpolating sequence (that is, $T(H_\infty) = l_\infty$) is that*

$$(C_1) \qquad \prod_{j \neq k} |(z_k - z_j)/(1 - \bar{z}_j z_k)| \geqq \delta > 0 \qquad (k = 1, 2, \cdots),$$

or equivalently,

(C_2) *there exist functions $f_k \in H_\infty$ such that*

(i) $f_k(z_i) = \delta_{ik}$,

(ii) $\| f \| \leqq 1/\delta \qquad (k = 1, 2, \cdots)$.

In this case each sequence $w = \{w_n\}$ in the unit sphere of l_∞ can be interpolated by an $f \in H_\infty$ with

$$\| f \| \leqq (2/\delta^5)(1 - 2 \log \delta). \tag{5}$$

We shall use the following notations.

$$B_n(z) = \prod_1^n (z - z_j)/(1 - \bar{z}_j z),$$

$$B_{nk}(z) = B_n(z)(1 - \bar{z}_k z)/(z - z_k) \qquad (k \leqq n),$$

$$b_{nk} = B_{nk}(z_k).$$

The equivalence of conditions (C_1) and (C_2) is easy to see. Indeed, if (C_1) is given, then the Blaschke product formed from the points $\{z_n\}$ with the point z_k deleted, divided by its value at z_k, may be taken as the function f_k.

Conversely, if (C_2) is given then $\| f_k/B_{nk} \| = \| f_k \| \leqq 1/\delta$. Evaluation at z_k gives $1/|b_{nk}| \leqq 1/\delta$ which implies (C_1).

Proof of Theorem 1. *Necessity.* The necessity of (C_2) was proven in the remarks preceding the statement of the theorem.

Sufficiency. Let $w = \{w_k\}$ be a bounded sequence ($|w_k| \leqq 1$ for all k) that we wish to interpolate at the points $\{z_k\}$. The function

$$h_n(z) = \sum_{k=1}^n w_k B_{nk}(z)/b_{nk}$$

solves the problem at the first n points. We rewrite this

$$h_n(z) = B_n(z) \sum_1^n (w_k/b_{nk})(1 - \bar{z}_k z)/(z - z_k).$$

Therefore the function

$$B_n(z)\{\sum_1^n (w/b_{nk})(1 - \bar{z}_k z)/(z - z_k) - g(z)\}$$

will also solve the problem, for any $g \in H_\infty$. We wish to choose g so as to minimize the norm in H_∞. Let

$$(6) \qquad m_n(w) = \inf_{g \in H_\infty} \| B_n(z)\{\sum (w_k/b_{nk})(1 - \bar{z}_k z)/(z - z_k) - g(z)\}\|_{H\infty}.$$

We must show that $m_n(w)$ is uniformly bounded in n and w (w in the unit ball of l_∞). Let us take (6) onto the boundary of the unit circle, and let ϕ denote the rational function given by the summation. Since $|B_n(e^{i\theta})| \equiv 1$ we have

$$m_n(w) = \inf_g \| \phi - g \|_{L\infty} \qquad (g \in H_\infty).$$

We now apply the duality relation (2), with $E = L_1$, $F = H_1$, $E^* = L_\infty$, $F^\perp = H_\infty$. We obtain

$$m_n(w) = \sup_f |(f, \phi)| \qquad (f \in H_1, \| f \|_1 = 1).$$

Evaluating (f, ϕ) by (1) we have

$$(7) \qquad m_n(w) = \sup_f |\sum_1^n (w_k/b_{nk}) f(z_k)(1 - |z_k|^2)|.$$

Since $\delta \leqq |b_{nk}| \leqq 1$ for all n, k we have

$$(8) \qquad \sup_f \sum_1^\infty |f(z_k)|(1 - |z_k|^2) \leqq \sup_n \sup_w m_n(w) \leqq \delta^{-1} \sup_f \sum_1^\infty |f(z_k)|(1 - |z_k|^2),$$

where $f \in H_1$, $\| f \|_1 = 1$, and w beolngs to the unit ball of l_∞.

Now if $f \in H_1$ has no zeros in the unit circle, then $f = g^2$ with $g \in H_2$. If f has zeros, then $f = Bf_1$, where B is a Blaschke product, f_1 has no zeros, $\| f \| = \| f_1 \|$, and $|f(z_k)| < |f_1(z_k)|$ for all k. Therefore the supremum on the right side of (8) is equal to

$$(9) \qquad \sup_g \sum |g(z_k)|^2 (1 - |z_k|^2) \qquad (g \in H_2, \| g \|_2 = 1).$$

We now show that the finiteness of (9) is equivalent to a weighted interpolation problem in H_2

LEMMA 1. *Let* $\{z_n\}$ *satisfy condition* (C_1) *of Theorem* 1. *Then the following two statements are equivalent:*

$$\sum |g(z_k)|^2 (1-|z_k|^2) \leqq M \|g\|^2 \tag{10}$$

for some constant M *and all* $g \in H_2$;

(11) *there is a constant* m *such that if* $\{w_k\}$ *is any square summable sequence, then there is a* $g \in H_2$ *with*

$$\|g\|^2 \leqq m \sum |w_k|^2 \text{ and } g(z_k)(1-|z_k|^2)^{\frac{1}{2}} = w_k \qquad (k = 1, 2, \cdots).$$

We shall show in fact that if (10) holds, then the constant m in (11) can be taken to be $(1/\delta^2)M$, while if (11) holds, then the constant M in (10) can be taken to be m.

Proof. The proof is similar to the proof given at the beginning of the sufficiency part of Theorem 1. Let $w = \{w_k\}$ be a sequence for which $\sum |w_k|^2 \leqq 1$. We wish to show that the problem of interpolating all such w in a uniformly bounded manner in H_2 (i.e. (11)) is equivalent to (10). Let

$$h_n(z) = \sum_1^n w_k B_{nk}(z)/b_{nk}(1-|z_k|^2)^{\frac{1}{2}}.$$

Then $h_n(z_k)(1-|z_k|^2)^{\frac{1}{2}} = w_k \quad (k = 1, 2, \cdots, n)$.

As before we modify h_n and arrive at (6), except that now $g \in H_2$ and the norm is the H_2 norm. The duality relation now leads to

$$m_n(w) = \sup_f \left| \sum_1^n (w_k/b_{nk}) f(z_k)(1-|z_k|^2)^{\frac{1}{2}} \right| \qquad (\|f\|_2 = 1). \tag{7'}$$

The analogue of (8) with both sides squared is:

$$\sup_f \sum |f(z_k)|^2 (1-|z_k|^2) \leqq [\sup_n \sup_w m_n(w)]^2 \leqq \delta^{-2} \sup_f \sum |f(z_k)|^2 (1-|z_k|^2), \tag{8'}$$

where $f \in H_2$, $\|f\|_2 = 1$, and w is in the unit ball of l_2. The number $[\sup_n \sup_w m_n(w)]^2$ will serve as the number m in (11), and the result follows.

To complete the proof of Theorem 1 it is now sufficient to establish (11), with $N = 2(1/\delta^4)(1 - 2\log\delta)$. This will be done in two lemmas, the first of which occurs in Carleson's paper.

LEMMA 2. *If* $\{z_k\}$ *satisfies condition* (C_1), *then*

$$\sum_{j=1}^{\infty} (1-|z_j|^2)(1-|z_k|^2)/|1-\bar{z}_j z_k|^2 \leqq 1-2\log\delta \qquad (k=1,2,\cdots). \tag{12}$$

Proof. The identity

$$|(z_k-z_j)/(1-\bar{z}_j z_k)|^2 = 1-(1-|z_j|^2)(1-|z_k|^2)/|1-\bar{z}_j z_k|^2$$

together with the inequality $1-b \leqq \exp(-b)$ and a little calculation shows that the series in (12), summed over $j \neq k$, is $\leqq -2\log\delta$, and the result follows from this.

LEMMA 3. *If* $\{z_k\}$ *satisfies condition* (C_1) *and if* $\{w_k\}$ *is any square summable sequence, then there is a* $g \in H_2$ *for which*

$$\|g\|^2 \leqq (2/\delta^4)(1-2\log\delta)\sum |w_k|^2$$

and

$$g(z_k)(1-|z_k|^2)^{\frac{1}{2}} = w_k \qquad (k=1,2,\cdots).$$

Proof. For this proof (f,g) will denote the inner product in H_2:

$$(f,g) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\bar{g}(e^{i\theta})\,d\theta \tag{13}$$

(in contrast to (1)). Let

$$g_{nk}(z) = (B_n(z)/(z-z_k))^2(1-|z_k|^2)^{\frac{3}{2}}$$

and

$$f_n(z) = \sum_1^n w_k g_{nk}(z)/b_{nk}^2.$$

Then $f_n(z_k)(1-|z_k|^2)^{\frac{1}{2}} = w_k$ $(k=1,2,\cdots,n)$. Moreover,

$$\|f_n\|^2 = (f_n, f_n) = \sum_{i,j=1}^{n} (w_i\bar{w}_j/(b_{ni}\bar{b}_{nj})^2)(g_i, g_j). \tag{14}$$

Here g_i stands for g_{n_i}. A calculation shows that

$$(g_i, g_j) = (1-|z_i|^2)^{\frac{3}{2}}(1-|z_j|^2)^{\frac{3}{2}}(1+z_i\bar{z}_j)/(1-z_i\bar{z}_j)^3.$$

Let $a_{ij} = (g_i, g_j)$. An easy calculation shows that $(1-|z_i|^2)^{\frac{1}{2}}(1-|z_j|^2)^{\frac{1}{2}} \leqq |1-\bar{z}_i z_j|$ for all i, j. Then by Lemma 2 we have:

$$\sum_j |a_{ij}| < 2\sum_j (1-|z_i|^2)(1-|z_j|^2)/|1-\bar{z}_i z_j|^2 \leqq 2(1-2\log\delta).$$

Since $a_{ji} = \bar{a}_{ij}$, the same bound holds for the sum on i. Therefore by (4) we have $\|f\|^2 \leqq (2/\delta^4)(1-2\log\delta)\sum |w_k|^2$, and the result follows.

This completes the proof of Theorem 1.

Remark 1. Condition (C_2) is conformally invariant and so Theorem 1 may be carried over to any simply connected domain. Thus in the strip $|y|<\sigma$ the points $z_k=k\pi$ satisfy condition (C_2) with the functions $f_k(z)=\sin(z-k\pi)/(z-k\pi)$. Here $1/\delta=\sup_k \sup_z |f_k(z)|=(e^{\sigma}-e^{-\sigma})/2\sigma$.

Remark 2. The functions f_n that were used in the proof of Lemma 3 are not the functions of minimal H_2 norm that solve the interpolation problem. It is a curious fact that if one uses the functions that must be minimal (their form is known, for example, from [13] Theorem 16), then one is led to a quadratic form that is much more difficult to handle than the one in (14). This form will be considered in the corollary to Theorem 3.

3. Interpolation by functions of class H_p. Let the sequence $\{z_n\}$ be given and let (s) denote the set of all sequences of complex numbers. For each p $(1\leqq p\leqq\infty)$ define a mapping $T_p: H_p\to(s)$ by:

$$T_pf=(f(z_1)(1-|z_1|^2)^{1/p}, f(z_2)(1-|z_2|^2)^{1/p}, \cdots).$$

THEOREM 2. *For each p $(1\leqq p\leqq\infty)$ the necessary and sufficient condition on $\{z_n\}$ in order that $T_pH_p=l_p$ is that $\{z_n\}$ satisfy condition (C_1), or equivalently,*

$(C_{2,p})$ *there exist functions $f_k\in H_p$ such that*

(i) $f_k(z_i)(1-|z_i|^2)^{1/p}=\delta_{ik}$

(ii) $\|f_k\|_p\leqq 1/\delta$.

The proof is very similar to that of Theorem 1 and so we shall merely sketch it. First, the equivalence of (C_1) and $(C_{2,p})$ is proven as before, using also the fact that $|f(z)|\leqq\|f\|_p/(1-|z|^2)^{1/p}$ for all $f\in H_p$ (see [8], p. 304).

One uses the closed graph theorem to show that if $l_p\subset T_pH_p$, then the interpolation can actually be done with uniformly bounded H_p norms. (If N is the subspace of functions vanishing on $\{z_n\}$, then a mapping is defined from l_p into H_p/N. This map has a closed graph.) This implies the necessity of $(C_{2,p})$.

It is easy to see that if $T_pH_p\subset l_p$, then T_p has a closed graph and is therefore a bounded operator (this was obvious for $p=\infty$).

LEMMA 4. *Let condition (C_1) be satisfied. Then $T_pH_p\subset l_p$ for a given value of p if and only if $l_q\subset T_qH_q$ for the conjugate index q $(1/p+1/q=1)$.*

The proof is similar to the proof of Lemma 1.

To complete the proof of Theorem 2 one proceds as before (going from (8) to (9)) to show that $T_pH_p \subset l_p$ if and only if $T_2H_2 \subset l_2$, which reduces the problem to Lemma 3. QED

Remark 3. In H_1 an interpolating function can be given explicitly. Let $\{w_k\}$ be the values to be interpolated, with $\sum |w_k|(1-|z_k|^2) < \infty$. Then

$$f(z) = \sum (B(z)/B'(z_k))\,[1/(z-z_k) - 1/(z-z_k^*)]w_k$$

will solve the problem. Here $B(z)$ denotes the Blaschke product formed from the z_k, and $z_k^* = 1/\bar{z}_k$. A simple calculation shows that the expression in square brackets has norm 1 in H_1 (since $|z-z_k| = |1-\bar{z}_k z|$ on $|z|=1$). Also, $\|B\| = 1$, and so under condition (C_1) we have an absolutely convergent series of unit vectors in H_1.

4. Interpolation in Hilbert spaces of analytic functions. The results in Section 2 on interpolation in the space H_2 suggest similar problems for other Hilbert spaces of analytic functions. By a Hilbert space H of analytic functions we mean a Hilbert space whose elements are analytic functions in a domain D, and in which evaluation at a point is a continuous linear functional on H. This means that to each point $\zeta \in D$ there corresponds a unique function $K_\zeta(z) \in H$ (the kernel function) such that

$$f(\zeta) = (f, K_\zeta) \tag{15}$$

for all $f \in H$. Thus for $\|f\| \leqq 1$ we have

$$|f(\zeta)| \leqq \|K_\zeta\| = (K_\zeta, K_\zeta)^{\frac{1}{2}} = (K_\zeta(\zeta))^{\frac{1}{2}}. \tag{16}$$

Equality is attained for $f = K_\zeta/\|K_\zeta\|$. It follows easily from (15) that $K_a(b) = \overline{K_b(a)}$ for all $a, b \in D$. Indeed, $K_a(b) = (K_a, K_b)$, $K_b(a) = (K_b, K_a)$, and these are conjugates of one another. For a general discussion of kernel functions and their properties see Aronszajn (*Proceedings of the Cambridge Philosophical Society*, vol. 39 (1943) and *Transactions of the American Mathematical Society*, vol. 68 (1950)).

We note the following examples.

I. H_2. Here $K_\zeta(z) = 1/(1-\bar{\zeta}z)$.

II. A_2. This is the space of functions $f(z) = \sum a_n z^n$, analytic for $|z| < 1$, with norm

$$\|f\|^2 = \tfrac{1}{\pi} \int\int_{|z|<1} |f(z)|^2\, dxdy = \sum |a_n|^2/(n+1) < \infty.$$

Here $K_\zeta(z) = 1/(1-\bar{\zeta}z)^2$.

III. H_2'. This is the space of functions $f(z) = \sum a_n z^n$ that are derivatives of functions of class H_2, with norm

$$\| f \|^2 = \tfrac{1}{\pi} \int\int |f(z)|^2 (1 - |z|^2)\, dxdy = \sum |a_n|^2/(n+1)(n+2) < \infty.$$

Here $K_\zeta(z) = 2/(1-\bar{\zeta}z)^3$.

IV. D (finite Dirichlet integral). These are the functions $f(z) = \sum a_n z^n$ analytic for $|z| < 1$, with $f(0) = 0$, whose derivative is in A_2. The norm is given by

$$\| f \|^2 = \tfrac{1}{\pi} \int\int |f'(z)|^2\, dxdy = \sum n |a_n|^2,$$

and $K_\zeta(z) = \log\{1/(1-\bar{\zeta}z)\}$.

V. Functions of exponential type in the upper half plane with

$$\| f \|^2 = \sup_y \tfrac{1}{2\pi} \int_{-\infty}^{\infty} |f(x+iy)|^2\, dx < \infty.$$

Here $K_\zeta(z) = i/(z+\bar{\zeta})$. By a Paley-Wiener theorem these functions are Fourier transforms of functions in $L_2(0, \infty)$.

VI. Entire functions of exponential type $\leqq \beta$ that belong to L_2 on the real axis. Here the norm is given by

$$\| f \|^2 = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} |f(x)|^2\, dx,$$

and $K_\zeta(z) = \{2 \sin \beta(z - \bar{\zeta})\}/(z - \bar{\zeta})$. By another Paley-Wiener theorem these functions are Fourier transforms of functions in $L_2(-\beta, \beta)$.

In this paper we consider only the first three of these spaces.

In H_2 we considered the problem of weighted interpolation: when will the sequence $\{z_n\}$ be such that $\{f(z_n)(1 - |z_n|^2)^{\frac{1}{2}}\}$ $(f \in H_2)$ is the most general sequence in l_2? In terms of the kernel function $K_\zeta(z)$ this becomes the following problem: for which sequences $\{z_n\}$ does $\{f(z_n)/(K_n(z_n))^{\frac{1}{2}}\}$ $(f \in H)$ represent the most general element in l_2? (We have written K_n for K_{z_n}.) In this form the problem can be posed for any Hilbert space of analytic functions. There are really two questions involved here: first, is the sequence always in l_2 for evey $f \in H$, and second, is every l_2 sequence obtained in this manner? Both of these questions can be formulated in an abstract Hilbert space as follows (see Bari [2]).

Let $\{x_n\}$ be a sequence of elements in a Hilbert space H. We shall say that $\{x_n\}$ is a *Bessel sequence* with bound M if

$$\sum |(x, x_n)|^2 \leqq M \| x \|^2 \tag{17}$$

for all $x \in H$. (If one merely assumes the convergence of the series on the left side of (17) for all $x \in H$, then one has a linear mapping from H to l_2. This mapping is easily seen to have a closed graph, which implies the existence of the constant M.)

We shall say that $\{x_n\}$ is a *Riesz-Fischer sequence* with bound m if to each sequence $\{c_n\} \in l_2$ there corresponds at least one $x \in H$ for which

$$(x, x_n) = c_n \ (n = 1, 2, \cdots) \text{ and } \| x \|^2 \leqq m \sum |c_n|^2. \tag{18}$$

Here also if one merely assumes the existence of $x \in H$ such that $(x, x_n) = c_n$ $(n = 1, 2, \cdots)$ for each square summable sequence $\{c_n\}$, then the existence of the constant m follows from the closed graph theorem. Indeed, if J is the subspace of H orthogonal to all the x_n, then one has a mapping from l_2 into H/J, and this mapping has a closed graph.

If H is a Hilbert space of analytic functions in a domain D, and if $\{z_n\}$ is a sequence of points in D, then we take for the sequence x_n the normalized kernel functions: $K_n(z)/\| K_n \|$, where $K_n(z) = K_{z_n}(z)$. The Bessel property then becomes the inequality

$$\sum |f(z_n)|^2 / K_n(z_n) \leqq M \| f \|^2 \qquad (f \in H),$$

while the Riesz-Fischer property is a statement about weighted interpolation: to each square-summable sequence $\{c_n\}$ there corresponds an $f \in H$ such that

$$f(z_n)/(K_n(z_n))^{\frac{1}{2}} = c_n \quad (n = 1, 2, \cdots), \quad \| f \|^2 \leqq m \sum |c_n|^2.$$

Nina Bari [2] shows that in an abstract Hilbert space these properties are equivalent to a number of other properties (her terminology is somewhat different from ours). We shall need only one such equivalence.

THEOREM 3. *Let $\{x_n\}$ be an infinite sequence of elements in a separable Hilbert space, and let A denote the inner product matrix $((x_i, x_j))$ $(i, j = 1, 2, \cdots)$. Then*

I. *$\{x_n\}$ is a Bessel sequence with bound M if and only if the matrix A is a bounded operator on l_2 with bound M.*

II. *$\{x_n\}$ is a Riesz-Fischer sequence with bound m if and only if the matrix A is bounded below on l_2 with bound m.*

The matrix A is said to be bounded below by m if $\| a \| \leqq m \| A_n a \|$ for all n-tuples $a = (a_1, a_2, \cdots, a_n)$ and all n. Here the norms are l_2 norms, and A_n is the $n \times n$ matrix (x_i, x_j) $(i, j = 1, 2, \cdots, n)$. This is equivalent to saying that $\| a \|^2 \leqq m(A_n a, a)$ for all n and a, and it is also equivalent to $\| A_n^{-1} \| \leqq m$ for all n.

Proof. I. Let $\{x_n\}$ be a Bessel sequence, let $\{c_n\} \in l_2$, let p be fixed, and let $y = c_1 x_1 + \cdots + c_p x_p$. On the one hand,

$$\| y \|^4 = |(y, y)|^2 = |\sum \bar{c}_n (y, x_n)|^2 \leqq \sum |c_n|^2 \sum |(y, x_n)|^2 \leqq M \| y \|^2 \sum |c_n|^2.$$

Dividing by $\| y \|^2$ we have: $\| y \|^2 \leqq M \sum |c_n|^2$. On the other hand,

$$\| y \|^2 = (y. y) = \sum c_i \bar{c}_j (x_i, x_j). \tag{19}$$

Thus the quadratic form in (19) is bounded by M, which implies that the matrix A is bounded by M.

Conversely, let A be bounded by M and let $\{e_n\}$ be a complete orthonormal set in H. We first show that there is a bounded operator $T: H \to H$, such that $Te_n = x_n$.

Indeed, let $x_i = \sum b_{ni} e_n$, and let T be the infinite matrix (b_{ni}). Then $T^*T = A$, and so $\| T \|^2 = \| T^* \|^2 = \| A \| = M$ (T^* is the conjugate transpose of T). Also, $Te_n = x_n$.

We now establish the Bessel property. Let $x \in H$ be given. Then $(x, x_n) = (x, Te_n) = (T^*x, e_n)$, and so

$$\sum |(x, x_n)|^2 = \| T^*x \|^2 \leqq M \| x \|^2.$$

II. Let $\{x_n\}$ have the Riesz-Fischer property, let $a_1, \cdots, a_n$ be given, and let $y \in H$ be the element of minimal norm such that $(y, x_i) = a_i$ $(i = 1, \cdots, n)$. Then y lies in the subspace spanned by $x_1, \cdots, x_n$ and therefore $y = c_1 x_1 + \cdots + c_n x_n$. Let $y_1 \in H$ be such that $(y_1, x_i) = a_i$ $(i \leqq n)$, $(y_1, x_i) = 0$ $(i > n)$, and $\| y_1 \|^2 \leqq m \sum |a_i|^2$. Then

$$\| y \|^2 \leqq \| y_1 \|^2 \leqq m \| a \|^2.$$

Also,

$$\bar{a}_i = (x_i, y) = \sum \bar{c}_j (x_i, x_j),$$

or more concisely, $\bar{a} = A_n \bar{c}$. Therefore

$$m \| a \|^2 \geqq (y, y) = \sum_{i,j} c_i \bar{c}_j (x_i, x_j) = (A_n \bar{c}, \bar{c}) = (\bar{a}, A_n^{-1} \bar{a}),$$

and so $\| A_n^{-1} \| \leqq m$.

Conversely, assume that $\| A_n^{-1} \| \leqq m$, and let $\{a_n\} \in l_2$ be given. To show that $\{x_n\}$ is Riesz-Fischer it will be sufficient to produce a sequence y_n such that

$$(y_n, x_i) = a_i \quad (i = 1, \cdots, n) \text{ and } \| y_n \|^2 \leqq m \sum | a_i |^2,$$

for by weak compactness a subsequence of y_n will converge to an element x satisfying (18).

Let $y_n = c_1 x_1 + \cdots + c_n x_n$. We wish to choose $\{c_i\}$ such that $a_i = (y_n, x_i)$ $(i = 1, \cdots, n)$. Just as before this means that $\bar{a} = A_n \bar{c}$, or $\bar{c} = A_n^{-1} \bar{a}$. Also,

$$\begin{aligned} \| y_n \|^2 = (y_n, y_n) = \sum c_i \bar{c}_j (x_i, x_j) &= (A_n \bar{c}, \bar{c}) \\ &= (\bar{a}, A_n^{-1} \bar{a}) \leqq m \| \bar{a} \|^2 = m \| a \|^2. \end{aligned}$$

This completes the proof.

We give an application of this theorem to the H_2 interpolation problem considered in Lemmas 1 and 3, in the case of exponential approach to the boundary. In this case we obtain a better constant than is given by Lemmas 1 and 3.

COROLLARY. *If the points $\{z_n\}$ satisfy the condition*

$$1 - | z_{n+1} | \leqq \alpha (1 - | z_n |)$$

for all n and some α $(0 < \alpha < 1)$, then

$$\sum | f(z_n) |^2 (1 - | z_n |^2) \leqq 2((1 + \alpha^{\frac{1}{2}})/(1 - \alpha^{\frac{1}{2}})) \| f \|^2$$

for all $f \in H_2$.

In Newman [11] (Theorem 2) it is shown that

$$1/\delta \leqq \prod_{n=1}^{\infty} ((1 + \alpha^n)/(1 - \alpha^n)),$$

where δ is the constant of condition (C_1) in Theorem 1. It can be shown that equality is attained here if $z_n = 1 - \alpha^n$ $(n = 1, 2, \cdots)$.

Proof. Since the kernel function for the space H_2 (the Szego kernel) is $1/(1 - \bar{\zeta} z)$, the corollary will follow from Theorem 3 if we can show that the matrix $A = \{a_{nm}\}$, where

$$a_{nm} = (1 - | z_n |^2)^{\frac{1}{2}} (1 - | z_m |^2)^{\frac{1}{2}} / (1 - \bar{z}_n z_m) \tag{20}$$

bounded above by $2(1 + \alpha^{\frac{1}{2}})/(1 - \alpha^{\frac{1}{2}})$. To do this we apply (4).

It follows from the hypothesis that $1-|z_{n+p}| \leqq \alpha^p(1-|z_n|)$. Also, $|1-\bar{z}_n z_m| \geqq 1-|z_{\min(n,m)}|$. Therefore

$$\sum_{n=1}^{\infty}|a_{nm}| = \sum_{1}^{m}|a_{nm}| + \sum_{m+1}^{\infty}|a_{nm}|$$

$$\leqq 2\sum_{1}^{m}((1-|z_m|)/(1-|z_n|))^{\frac{1}{2}} + 2\sum_{m+1}^{\infty}((1-|z_n|)/(1-|z_m|))^{\frac{1}{2}}$$

$$\leqq 2\sum_{1}^{m}\alpha^{(m-n)/2} + 2\sum_{m+1}^{\infty}\alpha^{(n-m)/2} < 2(1/(1-\alpha^{\frac{1}{2}}) + \alpha^{\frac{1}{2}}/1-\alpha^{\frac{1}{2}}).$$

5. Two special Hilbert spaces. In this section we consider the spaces A_2 and H_2' (see Examples II and III of Section 4). We shall need the following result of Schur [14].

THEOREM (Schur). *Let $A=(a_{ij})$ and $B=(b_{ij})$ be two Hermitian, positive definite $n \times n$ matrices of complex numbers. Let α_1, α_2 be the smallest and largest eigen-values of A, and let $\beta_1 = \inf b_{ii}$, $\beta_2 = \sup b_{ii}$ $(i=1,\cdots,n)$. Then $C=(a_{ij}b_{ij})$ is positive definite and all eigen-values of C lie between $\alpha_1\beta_1$ and $\alpha_2\beta_2$.*

Since the smallest and largest eigen-values are respectively the infimum and supremum of $\|Cx\|$ for $\|x\|=1$, it follows that

$$\|C\| \leqq \beta_2\|A\| \text{ and } \|C^{-1}\| \leqq (1/\beta_1)\|A^{-1}\|.$$

Let $\{z_n\}$ be a sequence of points in the unit circle and let H be a Hilbert space of analytic functions in the unit circle with kernel function $K_\zeta(z)$. We consider the unit vectors $K_n(z)/\|K_n\|$. Using the reproducing property (15) of the kernel function the inner product matrix of Theorem 3 becomes $(K_j(z_i)/\|K_i\|\,\|K_j\|)$. In particular, in H_2, A_2, and H_2' the general element of this matrix is:

(21) $$H_2\colon\ (\delta_i\delta_j)^{\frac{1}{2}}/(1-\bar{z}_i z_j),$$

(22) $$A_2\colon\ \delta_i\delta_j/(1-\bar{z}_i z_j)^2,$$

(23) $$H_2'\colon\ (\delta_i\delta_j)^{\frac{3}{2}}/(1-\bar{z}_i z_j)^3.$$

Here $\delta_i = 1-|z_i|^2$.

Finally, we introduce the operator T that associates to each $f \in H$ the weighted sequence of values

(24) $$Tf = (f(z_1)/\|K_1\|, f(z_2)/\|K_2\|, \cdots).$$

THEOREM 4. *Let $\{z_n\}$ be a sequence of points in the unit circle*

satisfying condition (C_1) *of Theorem* 1, *and let* $\delta_n = 1 - |z_n|^2$. *Then* $TA_2 = TH_2' = l_2$. *More precisely,*

$$\begin{aligned} &\Sigma |f(z_n)|^2 \delta_n^2 \leqq (1 - 2\log\delta)\|f\|^2 \quad (\text{all } f \in A_2) \\ &\Sigma |f(z_n)|^2 \delta_n^3 \leqq 2(1 - 2\log\delta)\|f\|^2 \quad (\text{all } f \in H_2'), \end{aligned} \tag{25}$$

and, to each square summable sequence $\{c_n\}$ *there corresponds an* $f \in A_2$ *and a* $g \in H_2'$ *such that*

$$f(z_n)\delta_n = g(z_n)\delta_n^{\frac{3}{2}} = c_n,$$
$$\|f\|^2 \leqq m \Sigma |c_n|^2, \qquad \|g\|^2 \leqq m \Sigma |c_n|^2.$$

Here the norms are taken respectively in A_2 and in H_2', and the number m ins the smallest admissible bound for the H_2 interpolation problem (by Lemma 3, $m \leqq (2/\delta^4)(1 - 2\log\delta)$).

Proof. It follows from (4) and Lemma 2 that the matrices (22) and (23) are each bounded by $(1 - 2\log\delta)$. Equations (25) now follow from Theorem 3.

Equations (26) follow from Schur's theorem and Theorem 3, since the elements in (22) and (23) are respectively the squares and the cubes of the elements in (21), and the diagonal elements are all equal to one. This completes the proof.

We shall need an analogue of Schwarz's lemma for the spaces A_2 and H_2'.

LEMMA 5. *Let* a, b *be two points in the unit circle and let*

$$d = (1 - |a|^2)(1 - |b|^2)/|1 - \bar{a}b|^2.$$

Then

$$|f(b)| \leqq (\|f\|/(1 - |b|^2))(|a - b|/|1 - \bar{a}b|)(1 + d)^{\frac{1}{2}}$$

for all $f \in A_2$ *for which* $f(a) = 0$. *Also*

$$|f(b)| \leqq (2\|f\|/(1 - |b|^2)^{\frac{3}{2}})(|a - b|/|1 - \bar{a}b|)(1 + d + d^2)^{\frac{1}{2}}$$

for all $f \in H_2'$ *for which* $f(a) = 0$. *Equality is possible in both cases.*

Proof. Let H denote either of the spaces A_2 or H_2', let $K_\zeta(z)$ be its kernel function and let J be the subspace of all functions vanishing at a. Then J is itself a Hilbert space of analytic functions with the kernel function

$$\phi_\zeta(z) = K_\zeta(z) - cK_a(z), \tag{27}$$

where $c = K_\zeta(a)/K_a(a)$. Indeed, $\phi_\zeta(a) = 0$, so for each fixed ζ, $\phi_\zeta \in J$. Also, if $f \in J$, then

$$(f, \phi_\zeta) = f(\zeta) - \bar{c}f(a) = f(\zeta)$$

and so ϕ_ζ reproduces the values of all $f \in J$. But the kernel function is always unique, and so ϕ_ζ is it.

Therefore for $f \in J$ we have:

$$|f(b)|^2 = |(f, \phi_b)|^2 \leqq \|f\|^2 (\phi_b, \phi_b) = \|f\|^2 \phi_b(b)$$
$$= \|f\|^2 (K_b(b) - |K_a(b)|^2 / K_a(a)).$$

Substituting the specific kernel functions for A_2 and H_2' we obtain the lemma.

We shall say that a sequence of points $\{z_n\}$ in the unit circle is *weakly separated* if there is a constant $c > 0$ such that

$$|z_i - z_j| \geqq c \max(\delta_i, \delta_j) \tag{28}$$

for all i, j (that is, the distance between points is greater than a constant multiple of the distance to the boundary). It is easy to see that if the z_n all lie on one radius, then this is equivalent to exponential approach to the boundary: $\limsup \delta_{n+1}/\delta_n < 1$.

Weak separation is equivalent to the following condition:

$$|z_i - z_j| \geqq c_1 |1 - \bar{z}_i z_j| \tag{29}$$

for some constant $c_1 > 0$ and all i, j. Indeed, writing a, b for z_i, z_j, we have $1 - |a|^2 \leqq 2(1 - |a|) \leqq 2|1 - \bar{a}b|$, and so (29) implies (28).

Conversely, a little calculation shows that if $|a| \leqq |b| < 1$ and if $x = |a - b|/(1 - |a|^2)$, then

$$|a - b|/|1 - \bar{a}b| \geqq x/(1 + x).$$

Therefore, (28) implies (29).

We now show that weak separation is a necessary condition for weighted interpolation.

THEOREM 5. *Let H be either of the spaces A_2, H_2', and let T be the operator in equation* (24). *If $l_2 \subset TH$, then the points $\{z_n\}$ are weakly separated. Furthermore, if the $\{z_n\}$ are weakly separated, then $TH \subset l_2$. More precisely, if* (28) *holds and if $b = \min(c, 1)$, then*

$$\begin{aligned} &\sum |f(z_n)|^2 \delta_n^2 \leqq (2/b)^2 \|f\|^2 && (\text{all } f \in A_2), \\ &\sum |f(z_n)|^2 \delta_n^3 \leqq 2(2/b)^2 \|f\|^2 && (\text{all } f \in H_2'). \end{aligned} \tag{30}$$

Proof. If $l_2 \subset TH$, then there is a constant m such that each element in the unit ball of l_2 can be obtained by weighted interpolation from an $f \in H$

with $\|f\| \leqq m$ (see the beginning of the proof of Theorem 2). Therefore, if i, j are given ($i \neq j$), then there is an $f \in H$, $\|f\| \leqq m$, such that

$$f(z_i)/(K_i(z_i))^{\frac{1}{2}} = 1, \qquad f(z_j) = 0.$$

(We could require $f(z_n) = 0$ $(n \neq i)$, but we shall only use the values of f at z_i, z_j.) Applying Lemma 5 with $a = z_i$, $b = z_j$ we have in A_2:

$$1/(1 - |z_i|^2) \leqq (m/(1 - |z_i|^2))(|z_i - z_j|/|1 - \bar{z}_i z_j|)(1 + d)^{\frac{1}{2}}.$$

Therefore

$$|z_i - z_j|/|1 - \bar{z}_i z_j| \geqq \text{const} > 0, \tag{31}$$

and this is equivalent to weak separation. Starting with the space H_2' we obtain (31) (with a different constant) in exactly the same way.

We now turn to the second part of the theorem. The points $\{z_n\}$ are given to be weakly separated and we must establish (30).

Let $b = \min(c, 1)$. It follows from (28) that the circles C_i: $|z - z_i| \leqq \frac{1}{2} d\delta_i$ are disjoint and are contained inside the unit circle. Also, $|f(z)|^2$ is a subharmonic function and so

$$|f(z_i)|^2 \delta_i^2 \leqq (2/b)^2 \frac{1}{\pi} \iint_{C_i} |f(z)|^2 \, dxdy \tag{32}$$

for all f analytic in the unit circle. If $f \in A_2$ we sum (32) on i and obtain the first equation of (30).

For $f \in H_2'$ we first multiply both sides of (32) by δ_i, and note the inequality: $1 - |z_i|^2 \leqq 2(1 - |z|^2)$ for $z \in C_i$. Indeed, the right side is smallest for $|z| = |z_i| + \frac{1}{2} d\delta_i \leqq |z_i| + \frac{1}{2}\delta_i$, and for this value the inequality may be verified. Now summing on i we obtain the second equation of (30). This completes the proof.

COROLLARY. *If the points $\{z_n\}$ all lie on one radius, then a necessary and sufficient condition for weighted interpolation ($l_2 \subset TH$) in either of the spaces A_2, H_2' is that the z_n approach the boundary exponentially. In this case, one has $l_2 = TH$.*

This follows at once from Theorems 4 and 5.

Remark 4. It can be shown that if $f \in H_2$ and $\phi(z) = 1/(1 - z)^{\frac{1}{2}}$ then $f_1 = \phi f \in A_2$, and $f_2 = \phi^2 f \in H_2'$. Hence if $f(z_n)\delta_n^{\frac{1}{2}} = c_n$, then $f_1(z_n)\delta_n = f_2(z_n)\delta_n^{\frac{3}{2}} = c_n$ $(n = 1, 2, \cdots)$. Thus if one can solve the H_2 problem explicitly for the case when the points z_n all lie on one radius, then one also has an explicit solution of the problem in the spaces A_2 and H_2'.

Finally, we would like to discuss a result of Epstein, Greenstein, and Minker [4]. They consider the space H of functions analytic in the strip D: $|y| < \sigma$ for which the area norm is finite

$$\|f\|^2 = \int\int_D |f(z)|^2 \, dxdy < \infty. \tag{33}$$

This is a Hilbert space with the kernel function

$$K_\zeta(z) = (\pi/16\sigma^2)\{\operatorname{sech}(\pi(z-\bar{\zeta})/4\sigma)\}^2.$$

They consider interpolation at the points $\{k\pi\}$ $(-\infty < k < \infty)$ and they show by Fourier analysis that the necessary and sufficient condition on a sequence $\{c_n\}$ that it be the set of values of an $f \in H$ at these points is that $\sum |c_n|^2 < \infty$. Since $K_x(x) = \text{const.}$ for x real, their problem can be considered as a problem in weighted interpolation.

To put the problem into our framework we consider the mapping

$$z = \phi(w) = \tanh(\pi w/4\sigma)$$

$(w \in D, |z| < 1)$ from the domain D to the unit circle. The points $w_n = n\pi$ go over into the points $z_n = \tanh(n\pi^2/4\sigma)$. These points lie on the real axis $(-1 < z_n < 1)$, and

$$1 - |z_n|^2 = \{\operatorname{sech}(n\pi^2/4\sigma)\}^2.$$

Thus $z_n \to 1$ exponentially as $n \to \infty$, and $z_n \to -1$ exponentially as $n \to -\infty$.

An application of Theorem 3 (just as in the corollary to Theorem 3) shows that

$$\sum_{-\infty}^{\infty} |f(z_n)|^2 (1 - |z_n|^2) \leqq (8 \coth(\pi^2/8\sigma)) \|f\|^2_{H_2}$$

for all $f \in H_2$. From Lemma 1, Remark 1, and Theorem 4 we obtain similar results for the values of functions in A_2 at these points. Finally, to go back to the space H in the strip it is necessary to "transplant" the functions of A_2 and multiply by the derivative of the mapping function. Thus, if $f \in A_2$, then

$$g(w) = f(\phi(w))\phi'(w)$$

is in H, and $g(w_n) = (\pi/4\sigma) f(z_n)(1 - |z_n|^2)$.

The theorem of [4] follows from this, and one could obtain a bound for the norm of the minimal interpolating function. The method just presented will establish a similar theorem for certain sets of points $\{w_n\}$ that are not in arithmetic progression. Using the corollary to Theorem 5 one can obtain the following result.

THEOREM 6. *A sequence of real numbers* $\{w_n\}$ *has the property that* $TH = l_2$ *if and only if there is a number* d *such that for all* n, m *one has*

$$|w_n - w_m| \geqq d > 0.$$

Here T is the operator: $Tf = (f(w_1), f(w_2), \cdots)$, and H is the Hilbert space of (33).

NEW YORK UNIVERSITY,
UNIVERSITY OF MICHIGAN, NEW YORK UNIVERSITY.

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REPRESENTATION OF A SEMIGROUP BY TRANSFORMATIONS ACTING TRANSITIVELY ON A SET.*

By E. J. Tully, Jr.[1]

A *representation* of a semigroup S will mean here a homomorphism of S into the semigroup of all transformations of some set M into itself. This concept may be identified in an obvious way with that of a *set with semigroup of operators*, which will also be called an *operand* over a semigroup, symbolized by M_S. By this we mean a set M together with a semigroup S and a mapping $(x, a) \to xa$ of $M \times S$ into M, such that $(xa)b = x(ab)$ for all $x \in M$, $a, b \in S$.

The theory of representations (by transformations) of groups has been extensively studied; accounts of this theory are given in [2] and [4]. The corresponding theory for semigroups has been studied relatively little. Stoll [6] has studied the representations of finite simple semigroups. He has also introduced the concept of transitive system (which we shall call *strictly cyclic operand* or *strictly cyclic representation*). By this is meant an operand M_S such that $xS = M$ for some $x \in M$, where $xS = \{xa : a \in S\}$. Thurston [8] has studied the *congruence relations on an operand* M_S (i.e., equivalence relations σ on M such that $x\sigma y$ implies $xa\sigma ya$ for all $x, y \in M, a \in S$), calling them S-equivalences, and calling M an S-set.

The present paper determines those representations of a semigroup S which are strictly cyclic (thus generalizing [6]), and more particularly those which are *transitive* (i.e., in which $xS = M$ for all $x \in M$; not the same thing as transitive system in the sense of Stoll) or 0-*transitive* (i.e., such that $|M| \geqq 2$ and $xS = M$ for all $x \in M$ except one x for which $xS = x$). In particular we shall obtain conditions for S to have *faithful* (i.e., 1-1) representations of the types considered.

Section 1 deals with strictly cyclic representations of a semigroup S. We show that all such representations arise (essentially) by letting S act by right multiplication on the set of σ-classes where σ is a right congruence on S such that $ea\sigma a$ (some $e \in S$, all $a \in S$) (1.1). The representation is faithful if and only if σ contains (i.e., is less fine than) no (2-sided) congruence except the identity relation (1.2). A necessary and sufficient condition for

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the existence of a faithful strictly cyclic representation (1.3) is an immediate corollary; this condition is very weak since it is satisfied by any semigroup with identity (1.4).

Section 2 shows that all transitive representations arise by letting S act on the σ-classes where σ is a right congruence, $ea\sigma a$ (some $e \in S$, all $a \in S$) and each σ-class meets each right ideal (2.1). A *n.a.s.c.* for the existence of a faithful transitive representation follows immediately (2.2). Similar but more complicated conditions are obtained for 0-transitive representations. As applications of these results, we show among other things that a commutative semigroup S has a faithful transitive representation if and only if S is a group (2.4) and (in Section 4) that the free semigroup of rank n has such a representation if and only if $n \geqq 2$.

Section 3 studies [0-] transitive representations of semigroups with certain finiteness conditions (viz., the existence of a minimal right ideal for the case of transitive representations, and minimum condition on principal right ideals for the case of 0-transitive representations). The main result gives a *n.a.s.c.* not involving right congruence relations for the existence of a faithful [0-] transitive representation (3.4 and 3.4′).

Section 4 gives some examples.

Notation. Transformations will be written on the right. If A and B are sets, $A - B$ denotes $\{x: x \in A, x \notin B\}$. $|A|$ denotes the cardinal of A. $A \subseteq B$ means "A is a subset of B." $A \subset B$ means "A is a proper subset of B." $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ are used in the sense of Green [3] to denote certain equivalence relations on a semigroup.

1. Strictly cyclic representations. We say that two operands M_S and N_S over the same semigroup are *isomorphic* (and we write $M_S \cong N_S$) if there is a 1-1 correspondence π between M and N with $(xa)\pi = (x\pi)a$ for all $x \in M$, $a \in S$. We call an element u of M a *strict generator* if $uS = M$. Thus M_S is strictly cyclic if and only if M has a strict generator.

If σ is a congruence relation on M_S, we can let S act on the set M/σ of σ-classes by defining $X \circ a$ ($X \in M/\sigma, a \in S$) to be the (unique) σ-class containing $Xa = \{xa: x \in X\}$. This gives a new operand which we denote by M_S/σ.

If S is a semigroup, we denote by S_S the operand in which S acts on itself by right multiplication. It is clear that the congruences on the operand S_S are precisely the right congruences on the semigroup S. Thus if σ is a right congruence on the semigroup, we can form the operand S_S/σ.

We call $e \in S$ a left identity modulo a right congruence σ if $ea\sigma a$ for all $a \in S$.

The following is the generalization to semigroups of the basic theorem on transitive representations of groups.

PROPOSITION 1.1. *Let u be a strict generator of M_S. Define a relation σ on S by: $a\sigma b$ if and only if $ua = ub$. Then σ is a right congruence, there is a left identity modulo σ, and $M_S \cong S_S/\sigma$.*

Proof. Suppose $a\sigma b$. Then $ua = ub$. Therefore $uac = ubc$ for all $c \in S$. This means that $ac\sigma bc$, so that σ is a right congruence. Let e be any solution of $ue = u$ (which exists because u is a strict generator). Then for $a \in S$, $uea = ua$ so that $ea\sigma a$. This shows that e is a left identity modulo σ. Now define a 1-1 mapping π of M onto S/σ by: $x\pi = \{a: ua = x\}$. Then it is easy to see that $(xa)\pi = (x\pi)a$, so that π is an isomorphism.

Note. Conversely, it is easy to see that if σ is a right congruence on S and there is a left identity modulo σ, then S_S/σ is strictly cyclic (each σ-class which contains a left identity modulo σ being a strict generator of S/σ).

We define the *kernel-congruence* κ of an operand M_S by: $a\kappa b$ if and only if $xa = xb$ for all $x \in M$. κ is a congruence on S. κ is the identity relation if and only if M_S is faithful.

PROPOSITION 1.2. *Let σ be a right congruence on a semigroup S, e a left identity modulo σ, κ the kernel-congruence determined by the operand S_S/σ. Then $a\kappa b$ if and only if $ca\sigma cb$ for all $c \in S$. Moreover κ is the largest left congruence (and therefore the largest 2-sided congruence) contained in σ. In particular, S_S/σ is faithful if and only if there is no left (equivalently, no 2-sided) congruence contained in σ except the identity relation. This is equivalent to the condition that $a = b$ whenever for all $c \in S$ $ca\sigma cb$.*

Proof. If $a\kappa b$, then $\Sigma \circ a = \Sigma \circ b$ ($\circ$ denoting the operation in S_S/σ) where Σ is any σ-class. This means that $ca\sigma cb$ for all $c \in S$, and the reasoning is reversible. Next we show that κ is contained in σ. Assume $a\kappa b$. Then $ea\sigma eb$, whence $a\sigma b$. Now suppose τ is any left congruence contained in σ. Then $a\tau b$ implies $ca\tau cb$ for all $c \in S$, so that $ca\sigma cb$. Hence $a\kappa b$. This shows that τ is contained in κ. The rest of the proposition is then clear.

COROLLARY 1.3. *A necessary and sufficient condition for a semigroup S to have a strictly cyclic faithful representation is that there exist a right congruence σ on S, such that there is a left identity modulo σ, and such that σ contains no left congruence except the identity relation.*

COROLLARY 1.4. *A sufficient condition for S to have a faithful strictly cyclic representation is that S have a left identity (and in this case S_S is strictly cyclic). If S is commutative then this condition is also necessary.*

2. Transitive representations of arbitrary semigroups. We call $X \subseteq M$ *invariant* if $XS \subseteq X$, where $XS = \{xa : x \in X, a \in S\}$. Thus, for M_S to be transitive (which means that each element of M is a strict generator) is equivalent to the assertion that the only invariant subsets of M are M and $\emptyset$. For M_S to be 0-transitive means that M contains an invariant element z and every $x \in M$ with $x \neq z$ is a strict generator. Let us call M_S the null operand of cardinal 2 if $|M| = 2$, $|MS| = 1$. Then it is easy to see that the following conditions are equivalent:

(1) M_S is either 0-transitive or the null operand of cardinal 2

(2) M has an invariant element z, and has no invariant subsets except M, $\{z\}$ and $\emptyset$.

We call M_S *primitive* if the only congruences on M are the identity relation and the universal relation. It is not difficult to show that if $|M| \geqq 3$ and M is primitive, then M is either transitive or 0-transitive.

Examples. Let M be any set. If S consists of all transformations on M, M_S is primitive and transitive. If $|M| \geqq 3$ and $S' = \{a \in S : |Ma| = 1\}$, then $M_{S'}$ is transitive but not primitive. If $|M| \geqq 2$, $z \in M$ and $T = \{a \in S : za = z\}$, M_T is primitive and 0-transitive. If $|M| \geqq 3$, $z \in M$ and $T' = \{a \in T : |(M - \{z\})S| = 1$, then $M_{T'}$ is 0-transitive but not primitive. Let M_S be any operand. Let $x \in M$, $X = xS \cup \{x\}$. Define σ on X_S by: $w \sigma y$ if and only if $w = y$ or $w, y \in Z = \{z \in X : x \notin zS\}$. Then X_S/σ is transitive or 0-transitive, according as $Z = \emptyset$ or $Z \neq \emptyset$.

PROPOSITION 2.1. *Let σ be a right congruence on S. The following are equivalent:*

(3) *S_S/σ is transitive*

(4) *For all $a, b \in S$ there exists $c \in S$ with $ac \sigma b$*

(5) *Each σ-class meets each right ideal of S.*

PROPOSITION 2.1′. *Let σ be as in 2.1. The following are equivalent:*

(6) *S_S/σ is 0-transitive or the null operand of cardinal 2*

(7) *One σ-class $R \neq S$ is a right ideal, and for all $a \in S - R$, $b \in S$ there exists $c \in S$ such that $ac \sigma b$*

(8) *One σ-class $R \neq S$ is a right ideal, and every σ-class meets every right ideal which is not contained in R.*

Proof. First note that for a σ-class A to be a strict generator of S/σ means that for all $a \in A$, $b \in S$ there exists $c \in S$ with $ac \sigma b$, or equivalently that every right ideal which meets A meets every σ-class. Hence (4) and (5) each assert that every σ-class is a strict generator, and this is equivalent

to (3). For 2.1′, note that a σ-class A is an invariant element of S/σ if and only if A is a right ideal. Thus (7) and (8) each assert that $|M| \geqq 2$, that R is an invariant element of S/σ, and that every element of S/σ other than R is a strict generator. This is equivalent to (6).

Note. Many results (like the preceding proposition) are stated in two forms (one for transitive, the other for 0-transitive, representations). Hereafter we shall prove only the primed proposition of each pair. The other may be proved by similar methods.

COROLLARY 2.2. *A necessary and sufficient condition for S to have a faithful transitive representation is that there exist a right congruence σ on S such that σ satisfies* (4) *and contains no left congruence* [*equivalently, no 2-sided congruence*] *except the identity relation and such that there is a left identity modulo σ.*

COROLLARY 2.2′. *A necessary and sufficient condition for S to have a faithful 0-transitive representation is that there exist a right congruence σ on S such that σ satisfies* (7) *and contains no left congruence* [*equivalently, no 2-sided congruence*] *except the identity relation and such that there is a left identity modulo σ and $|S/\sigma| \geqq 2$.*

Proof. Combine 1.1, 1.2 and 2.1′.

COROLLARY 2.3 [2.3′]. *A sufficient condition for S to have a faithful* [0-] *transitive representation is that S be right* [*zero*] *simple and have a left identity. (In this case S_S is* [0-] *transitive and faithful.)*

PROPOSITION 2.4 [2.4′]. *A necessary and sufficient condition for a commutative semigroup S to have a faithful* [0-] *transitive representation is that S be a group* [*to which a zero-element has been adjoined*].

Proof. Given a faithful 0-transitive representation, obtain σ by 2.2′. By commutativity σ is a 2-sided congruence. Hence, σ is the identity relation. Hence S contains an actual left identity, e, and is right 0-simple with 0-element z. Clearly, $S - \{z\}$ is an $\mathcal{H}$-class containing an idempotent (viz., e), and hence is a group, so that S is a group with zero. The converse follows from 2.3′.

3. Transitive representations of semigroups with finiteness conditions. If R is a right ideal in S, R_S denotes the operand in which S acts on R by ordinary right multiplication.

PROPOSITION 3.1 [3.1′]. *Let M_S be a* [0-] *transitive operand* [*with invariant element z*]. *Let R be a right ideal of S* [*such that $MR \neq \{z\}$*]. *Then $M_S \cong R_S/\sigma$ for some congruence σ on R_S.*

Proof. Let $u \in M$ be such that $uR \neq \{z\}$. Then u is a strict generator. Hence by 1.1 $M \cong S/\pi$ where π is given by: $a\pi b$ if and only if $ua = ub$. Let σ be π restricted to R. Define $\phi: S/\pi \to R_S/\sigma$ by $\Pi\phi = \Pi \cap R$, where Π is a π-class. Let $J = \{a \in S: ua = z\}$ be the (unique) π-class which is a right ideal. Then $R - J \neq \emptyset$. Hence by 2.1′ R meets every π-class. Hence ϕ is onto. It is then clear that ϕ is an isomorphism.

PROPOSITION 3.2 [3.2′]. *Let R be a [non-] minimal [principal] right ideal in S. Let σ be a congruence on S [such that all non-generators in R are contained in a single σ-class Σ]. Then R_S/σ is a [0-] transitive operand [having Σ as its (unique) invariant element].*

Proof. The set J of non-generators in R is the largest proper invariant subset of R_S. Thus R_S/σ has no invariant subsets between $\{\Sigma\}$ and the whole of R/σ, since Σ is the image of J under the natural mapping $R \to R/\sigma$. Clearly Σ is an invariant element of R/σ, so that R/σ is 0-transitive.

PROPOSITION 3.3 [3.3′]. *Let R and σ [and J] be as in 3.2 [3.2′]. Then the kernel-congruence κ of the operand R_S/σ is given by: $a\kappa b$ if and only if $ca\sigma cb$ for all $c \in R$. In particular R_S/σ is faithful if and only if for all $a, b \in S$ with $a \neq b$ there exists $c \in R$ with $ca \not\sigma cb$.*

Proof. Clear.

THEOREM 3.4. *Let R be a minimal right ideal in S. Then the following are equivalent:*

(9) *S has a faithful transitive representation*

(10) *$ca = cb$ for all $c \in R$ implies $a = b$*

(11) *$ca = cb$ for all $c \in K$ implies $a = b$, where K is the union of the minimal right ideals of S.*

Proof. (9) $\Rightarrow$ (10): Let M_S be faithful and transitive. Then $M \cong R_S/\sigma$, where σ is as in 3.1. Hence (10) follows from the last part of 3.3.

(10) $\Rightarrow$ (11): Clear, since $R \subseteq K$.

(11) $\Rightarrow$ (9): R_S is transitive by 3.2 (with $\sigma =$ the identity relation). Suppose R_S is not faithful. Then there exist $a, b \in S$ with $a \neq b$ and $ca = cb$ for all $c \in R$. By semigroup theory $K = SR$. Hence for all $d \in K$, there exists $f \in R$, $g \in S$ with $da = gfa = gfb = db$. This contradicts (11)..

THEOREM 3.4′. *Let S be a semigroup which satisfies the minimum condition on principal right ideals. Then S has a faithful 0-transitive representation if and only if there exists an $\mathfrak{R}$-class R of S, not a minimal right*

ideal, such that (*for* $a, b \in S$) $a = b$ *whenever for all* $c \in R$ *either* $ca = cb$ *or* $ca, cb \notin R$.

Proof. First let M_S be faithful and transitive. By 1.1 $M \cong S/\sigma$ for some right congruence σ on S. By 2.1′ one σ-class $\mathfrak{A}$ is a right ideal and $\mathfrak{A} \neq S$. Let R' be minimal in the class of all principal right ideals not contained in $\mathfrak{A}$. Let R be the $\mathfrak{R}$-class consisting of all generators of R'. By 3.1′ $M \cong R'_S/\sigma$ for some congruence σ on R'_S. By 2.1′ $\mathfrak{A} \cap R' \neq \emptyset$. Hence if R' were a minimal right ideal $\mathfrak{A}$ would contain R', contradicting the definition of R'. Moreover if $a \in (R' - R) - \mathfrak{A}$ then $\mathfrak{A}$ would not contain the right ideal $aS \cup \{a\}$ contradicting the minimality of R'. Thus we can apply 3.3′, so that for all $a, b \in S$ with $a \neq b$ there exists $c \in R'$ with $ca \not\sigma cb$. Hence $ca \neq cb$. Moreover ca and cb are not both in $\mathfrak{A}$ (since $\mathfrak{A}$ is contained in a single σ-class). Thus we have shown that R satisfies the condition stated in the theorem. Conversely, suppose such an $\mathfrak{R}$-class R exists. Let R' be the principal right ideal generated by R. Define σ on R' by: $a \sigma b$ if and only if either $a = b$ or $a, b \notin R$. By 3.2′ R'_S/σ is 0-transitive. By 3.3′ R'_S/σ is faithful. (Note that we used the minimum condition only for the proof of the necessity of the condition.)

4. Examples. We call S (right) transitive if S has a faithful transitive representation, and left transitive if S is anti-isomorphic to a transitive semigroup. It is clear that the results of Sections 2 and 3 concerning transitive semigroups can be applied to left transitive semigroups by interchanging "left" and "right."

By 2.4 any commutative semigroup which is not a group is neither left nor right transitive. Moreover if S has a right ideal R such that $ca = cb$ for all $a, b \in S$, $c \in R$ then (as an immediate consequence of 3.1 and 3.3) S is not transitive. In particular, a semigroup with a zero element is neither left nor right transitive. If S is completely simple in the sense of Rees [5] with structure matrix P then it is easy to show (by 3.4) that S is transitive if and only if P has two rows which are not left multiples of each other. 3.4 gives a rather explicit condition for any semigroup having a minimal right ideal to be transitive.

We now give examples not having minimal right ideals.

Example 1. The bicyclic semigroup, which consists of all pairs of non-negative integers with multiplication given by:

$$(i, j)(k, l) = (i + k - \min(j, k), j + k - \min(j, k)),$$

is transitive on both sides. The relation σ, given by: $(i, j) \sigma (k, l)$ if and only

if $j \neq l$, satisfies 1.3. (σ coincides with the relation $\mathfrak{L}$ in the sense of Green [3].)

Example 2. Let S be a semigroup containing at least two elements, T any semigroup, and P the free product (either with or without an identity element adjoined) of S and T. Then P is transitive on both sides.

Proof. Choose a fixed $a \in S$. Let Q be the set of all elements of P of one of the following forms: t, where $t \in T$; st, where $a \neq s \notin S$, $t \in T$; $\prod_{i=1}^{n} at_i \prod_{i=n+1}^{m} s_i t_i$, where $s_{n+1} \neq a$, $m = 2n$, $s_i \in S$, $t_i \in T$. Now, given any $p \in P$, write p (uniquely) as a product of elements belonging alternately to S and to T. From the left of the sequence thus obtained, delete repeatedly elements of Q until this is no longer possible. Call the resulting (possibly empty) sequence the normalized form of p. Define a right congruence σ on P by: $p\sigma q$ if and only if p and q have the same normalized forms. Then it is easy to see that σ satisfies 2.2.

In particular, the free semigroup of rank $n \geqq 2$ is transitive on both sides. (Recall that by 2.4 the free semigroup of rank 1 is not transitive.)

Example 3. Let $S = \{a\}$, $T = \{b\}$. Let P be the free product (with or without an identity) of S and T. Then P is neither left nor right transitive.

Proof. For $i = a$ or b, let $(i, n) \in P$ denote the product in P of n elements, alternately a and b, the first one being i. Suppose P is (say) right transitive. Let σ be as in 2.2. By (5) the σ-class containing a meets the right ideal $\{(a, n) : n > 1\}$. Let p be the least integer with $a\sigma(a, p)$ and $p > 1$. Similarly let q be the least integer with $b\sigma(b, q)$ and $q > 1$. Case I: one of p, q (say p) is even. Here $ab\sigma(a, p)$ so that $p = 2$. Hence $a\sigma(a, n)$ for every n. The σ-class containing b must meet the right ideal $\{(a, n) : n \geqq 1\}$. Hence $a\sigma b$. Hence P consists of a single σ-class, contradicting 2.2. Case II: p and q are both odd. Here it is easy to see that for all $x \in P$ we have $xa\sigma x(a, 1 + (p-1)(q-1))$ and yet $a \neq (a, 1 + (p-)(q-1))$. This contradicts the last sentence of 1.2.

Example 4. Let M be an infinite set. Let S be the Baer-Levi semigroup (given in [1]), which consists of all 1-1 transformations of M into M such that $|M - M\pi| = |M|$. S is obviously right transitive. We shall show that S is also left transitive.

Proof. Order M by the first ordinal whose cardinal is $|M|$. Then for $N \subseteq M$, $|N| = |M|$, it follows that N (ordered by restriction) has the same ordinal as M. Hence there is a (unique) order isomorphism π_N of N onto M.

Given $\alpha \in S$, $\pi_{M\alpha}$ is a permutation of M. Define σ on S by: $\alpha\sigma\beta$ if and only if $\pi_{M\alpha} = \pi_{M\beta}$. It can be shown that σ is a left congruence, that each σ-class meets each left ideal, and that given $\alpha, \beta \in S$ with $\alpha \neq \beta$, there exists $\gamma \in S$ with $\alpha\gamma$ and $\beta\gamma$ in different σ-classes.

Example 5. The direct product of any number of transitive semigroups is transitive.

Example 6. Let S be a semigroup. Let $P = (p_{ij})$ be an $m \times n$ matrix of elements of S. Let T be the Rees matrix semigroup, which consists of all $n \times m$ matrices having an element of S in one place and zeros elsewhere, multiplication being defined by $A \circ B = APB$ for $A, B \in T$. This semigroup was considered in [7] and [5] for the case where S is a group.

Two rows, i and j, of P will be called dependent if, for some $a, b \in S$, $p_{ik}a = p_{jk}b$ for all k. Now if S is right transitive and no two rows of P are dependent, T is right transitive. (For, given σ on S as in 2.2, identify two elements $A = (a_{ij})$ and $B = (b_{ij})$ of P whenever for all i, j either $a_{ij} = b_{ij} = 0$ or $a_{ij}\sigma b_{ij}$. This gives a right congruence having the desired properties.) On the other hand, if two rows of P are dependent, then T is not right transitive.

CALIFORNIA INSTITUTE OF TECHNOLOGY.

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ON DIFFERENTIALS IN FUNCTION FIELDS.*

By Oscar Zariski[1] and Peter Falb.

Introduction. In order to develop a theory of differentials in algebraic functions fields K, of one variable, over *imperfect* ground fields k, and—above all—in order to prove in this case the classical formula which relates the order of dx at a place of K/k to the order of the different D_x at that place, Chevalley, in his book on Algebraic Functions of one variable, defines differentials by means of repartitions in K/k. The object of this paper is to show that also in the case of imperfect ground fields it is possible to develop the theory by using the conventional (formal) definition of a differential, without having recourse to repartitions. The advantage of our approach is that it enables one to deal with differentials $dx_1dx_2\cdots dx_r$ in function fields K/k of r independent variables (where repartitions are not available if $r>1$), by identifying (in an obvious sense) such a differential with the differential dx_1 in the field $K/k(x_2,\cdots,x_r)$ (if the characteristic p of k is different from zero, our new ground field $k'=k(x_2,\cdots,x_r)$ will not be perfect, even if the original ground field k was algebraically closed.).

Our proofs are essentially based on methods which have been developed by one of us in a paper on simple points (see Zariski [1]), and, in particular, on the mixed Jacobian criterion for simple points, established in that paper. As in that paper, so also here, we fix a model (normal curve) of K/k, we observe that any such model is already defined over $k^p(u_1,\cdots,u_\rho)$, where the u_i form a finite set of p-independent elements of k, and we deal with derivations with respect to the "p-parameters" u_i. This allows us to define the notion of "uniformizing coördinates" (which have been defined previously only for absolutely simple points) and to establish a crucial criterion for uniformizing coördinates (see § 1, Theorem 1). The notion of the order of a differential dx at a place p is defined by using uniformizing coördinates (see § 2, Definition 2). The role of Theorem 1 in the proof of the main result concerning the different (§ 3, Theorem 2) is that it allows us to reduce at once the proof of Theorem 2 to the case of cyclic Galois extensions, of prime

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order. This cyclic case is treated in the last two sections of the paper, and here again the use of Theorem 1 is essential.†

We note that in the case of function fields K/k of r independent variables $x_1, \cdots, x_r$, where we take as new ground field the field $k(x_2, \cdots, x_r)$ (k-algebraically closed), the variables $x_2, \cdots, x_r$ play precisely the role of the p-parameters $u_1, \cdots, u_\rho$.

1. Uniformizing coordinates. Let V be an irreducible, r-dimensional affine variety, immersed in an affine space S_n and defined over a ground field k of characteristic $p \neq 0$. We assume that the function field K of V over k is a separable extension of k. Let $u_1, \cdots, u_\rho$ be a finite p-independent set of elements of k such that the ideal, $\mathfrak{d}(V)$, of V in $k[X_1, \cdots, X_n]$ has a basis $f_1(X), \cdots, f_N(X)$ of polynomials with coefficients in $k^p(u_1, \cdots, u_\rho)$; and let k_1 be a maximal subfield of k, containing k^p and such that $[k_1(u_1, \cdots, u_\rho) : k_1] = p^\rho$. We note that $k = k_1(u_1, \cdots, u_\rho)$. We will call $u_1, \cdots, u_\rho$ *p-parameters* of V/k, and k_1 a field of *p-constants* associated with the p-parameters $u_1, \cdots, u_\rho$. Any finite p-independent set U which contains a set U_0 of p-parameters of V/k is also a set of p-parameters of V/k, and if k_U is a field of p-constants associated with U, then we may take a suitable subfield of k_U as field of p-constants associated with U_0. This possibility of enlarging a set of p-parameters (and correspondingly shrinking the field of

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† *Note added by O. Zariski in galley proofs.* During my visit to Frankfurt in June, 1961, Dr. Ernst Kuntz has called my attention to his most recent paper "Differentialformen inseparabler algebraischen Funktionenkörper" (*Math. Zeitschr.*, vol. 76 (1961), pp. 56-74) in which an essentially similar treatment of differentials over imperfect ground fields is found. In many respects the work of Dr. Kuntz goes farther than ours, namely (1) he does not assume that K/k is separable and (2) he deals also with function fields of varieties of arbitrary dimension (this latter generalization has also been carried out by Dr. Falb in his unpublished Harvard dissertation; in their presentation of the behaviour of differential forms on complete algebraic models of K/k, both Kuntz and Falb follows the procedure given by me in my lecture notes "Introduction to the theory of Algebraic Surfaces," Harvard University, 1957-1958). On the other hand, our characterization of uniformizing coördinates (Theorem 1) is not in Kuntz's paper. As this theorem plays an essential role in our paper, our treatment differs from that given by Kuntz in a significant way.

The paper of Kuntz is followed immediately, in the same issue of the *Mathematische Zeitschrift*, by a paper of Hans-Joachim Nastold, entited "Zum Dualitätssatz in inseparablen Funktionen = körpern der Dimension 1" (pp. 75-84). In this paper Dr. Nastold, using the work of Kuntz, proves the duality theorem for inseparable function fields K/k of dimension 1. Thus, on the basis of these two papers by Kuntz and Nastold, we have now, also for inseparable function fields, a complete formulation of the Riemann-Roch theorem, in which the differentials (or—equivalently—the canonical divisor class) appear in their conventional form (and not as repartitions).

p-constants) allows us to deal with the case of a finite set of varieties over k; i.e., there will then exist elements $u_1, \cdots, u_\rho$ and a field k_1 such that $\{u_1, \cdots, u_\rho\}$ is a set of p-parameters having k_1 as associated field of p-constants, for each variety of the set. For the moment, however, we shall assume that the p-parameters $u_1, \cdots, u_\rho$ and the field k_1 of p-constants have been fixed. As we shall consider derivations of K and of other fields or rings containing k, we stipulate now that *all derivations will be tacitly assumed to be trivial on* k_1 (but not necessarily on k).

Let W be an irreducible, s-dimensional subvariety of V/k, let $\mathfrak{O}$ be the local ring of W on V and let $\mathfrak{M}$ be the maximal ideal of $\mathfrak{O}$. We shall use the following notations:

$\mathfrak{D}(K)$ = space of derivations of K (over k_1, by our convention);

$\mathfrak{D}(W)$ = set of derivations D of K which are regular on W, i.e., such that $D\mathfrak{O} \subset \mathfrak{O}$;

$\mathfrak{D}^*(W)$ = space of local W-derivations D^* on V, i.e., derivations D^* of $\mathfrak{O}$, with values in the residue field $k(W)$ of $\mathfrak{O}$;

$\bar{\mathfrak{D}}(W)$ = space of derivations of $k(W)$.

$\mathfrak{D}(K)$ and $\mathfrak{D}(W)$ are vector spaces over K and $k(W)$ respectively, $\mathfrak{D}(K)$ being of dimension $r+\rho$ (since K/k is separable and $[k: k_1] = p^\rho$); $\mathfrak{D}(W)$ is an $\mathfrak{O}$-module while $\mathfrak{D}^*(W)$ is (canonically) a $k(W)$-vector space ([1], p. 34). If W is simple on V, then $\dim \mathfrak{D}^*(W) = r+\rho$ as a consequence of the Jacobian criterion for simple subvarieties ([1], p. 39). We have a mapping $D \to D^*$ of $\mathfrak{D}(W)$ into $\mathfrak{D}^*(W)$ given by $D^*z = \overline{Dz}$, $z \in \mathfrak{O}$ (where a bar above an element of $\mathfrak{O}$ denotes the $\mathfrak{M}$-residue of that element). Every element $\bar{D}$ of $\bar{\mathfrak{D}}(W)$ can be lifted to an element $\bar{D}^*$ of $\mathfrak{D}^*(W)$ by setting $\bar{D}^*z = \bar{D}\bar{z}$, $z \in \mathfrak{O}$. The mapping $\bar{D} \to \bar{D}^*$ is clearly an injection.

DEFINITION 1. *$\rho+r$ elements, $\xi_1, \cdots, \xi_{\rho+r}$, of $\mathfrak{O}$ are uniformizing coordinates of W if there exist $\rho+r$ derivations D_i of K, regular on W, such that $D_i\xi_j = \delta_{ij}$ $(i, j = 1, \cdots, \rho+r)$.*

PROPOSITION 1. *If W is simple on V, then there exist uniformizing coordinates of W.*

Proof. Let $x_i = \bar{X}_i = \mathfrak{J}(V)$-residue of X_i and let $\eta^1, \cdots, \eta_{\rho+n}$ stand for $u_1, \cdots, u_\rho, x_1, \cdots, x_n$ in some order (where $u_1, \cdots, u_\rho$ are the p-parameters). We may then assume that the Jacobian determinant

$$J(f_1, \cdots, f_{n-r}; \eta_{\rho+r+1}, \cdots, \eta_{\rho+n})$$

is not zero on W (i.e., is not an element of $\mathfrak{M}$), since the rank of the Jacobian matrix $J(f_1, \cdots, f_N; \eta_1, \cdots, \eta_{\rho+n})$ is $n-r$ ([1] p. 39). It follows that there exist $D_1, \cdots, D_{\rho+r}$ in $\mathfrak{D}(K)$ such that $D_i\eta_j = \delta_{ij}$ $(i, j = 1, 2, \cdots, \rho+r)$ and that these derivations are uniquely determined. We have, for

$i = 1, \cdots, \rho+r$; $\nu = 1, \cdots, n-r$:

$$0 = D_i f_\nu = \frac{\partial f_\nu}{\partial \eta_i} + \sum_{\mu=1}^{n-r} \frac{\partial f_\nu}{\partial \eta_{\rho+r+\mu}} D_i\eta_{\rho+r+\mu}.$$

For fixed i, these equations, viewed as equations in the $D_i\eta_{\rho+r+\mu}$, have determinant which does not belong to $\mathfrak{M}$. Hence $D_i\eta_j \in \mathfrak{O}$ $(j = 1, \cdots, \rho+n)$. But $\mathfrak{O}$ is a quotient ring of $k_1[\eta_1, \cdots, \eta_{\rho+n}]$; therefore, the D_i are regular on W. Consequently $\eta_1, \cdots, \eta_{\rho+r}$ are uniformizing coordinates of W.

COROLLARY. *If W is simple on V, then $\mathfrak{D}(W)$ is a free $\mathfrak{O}$-module of rank $\rho+r$, and if $\xi_1, \cdots, \xi_{\rho+r}$ are uniformizing coordinates of W and $D_1, \cdots, D_{\rho+r}$ are as in Definition 1, then the D_i form a free basis of $\mathfrak{D}(W)$ over $\mathfrak{O}$. The mapping $D \to D^*$ of $\mathfrak{D}(W)$ into $\mathfrak{D}^*(W)$ given by $D^*z = \overline{Dz}$, $z \in \mathfrak{O}$, is a surjection, with kernel $\mathfrak{M}\mathfrak{D}(W)$.*

Proof. It is clear that the derivations $D_1, \cdots, D_{\rho+r}$ of Definition 1 are linearly independent over $\mathfrak{O}$ and therefore form a basis of $\mathfrak{D}(K)$ (over K). If $D \in \mathfrak{D}(W)$, then $D = \sum \lambda_i D_i$, $\lambda_i \in K$, and since $\lambda_i = D\xi_i$, it follows that the λ_i are in $\mathfrak{O}$. Hence the D_i form a free $\mathfrak{O}$-basis of $\mathfrak{D}(W)$. The second assertion follows from the fact that rank $\mathfrak{D}(W)/\mathfrak{O} = \dim \mathfrak{D}^*(W)/k(W)$ $(= \rho+r)$.

PROPOSITION 2. *Let $\eta_1, \cdots, \eta_{\rho+r}$ be uniformizing coordinates of W and let $D'_1, \cdots, D'_{\rho+r}$ be derivations of K, regular on W, such that $D'_i\eta_j = \delta_{ij}$. Then, given $\rho+r$ elements $\xi_1, \cdots, \xi_{\rho+r}$ in $\mathfrak{O}$, a necessary and sufficient condition that these elements be uniformizing coordinates of W is that the determinant $|D'_i\xi_j|$ does not belong to $\mathfrak{M}$.*

Proof. Assume that the ξ_i are uniformizing coordinates of W and let $D_1, \cdots, D_{\rho+r}$ be as in Definition 1. Since the D'_i form a basis of $\mathfrak{D}(K)$ over K (actually a free $\mathfrak{O}$-base of $\mathfrak{D}(W)$, by the previous corollary) we have $D_i = \sum_{j=1}^{\rho+r} \lambda_{ij} D'_j$ with $\lambda_{ij} \in K$ (in fact, $\lambda_{ij} \in \mathfrak{O}$), and $\lambda_{ij} = D_i\eta_j$. Applying D_i to ξ_q, we find that $\delta_{iq} = \sum_j (D_i\eta_j)(D'_j\xi_q)$, which shows that $\| D_i\eta_j \| \cdot \| D'_j\xi_q \|$ is the unit matrix. Since the determinant of each of these matrices is in $\mathfrak{O}$, each determinant is a unit in $\mathfrak{O}$.

Conversely, if $|D'_i\xi_j| \notin \mathfrak{M}$, then the matrix $\|D'_i\xi_j\|$ (whose entries are in $\mathfrak{O}$, as $D'_i \in \mathfrak{D}(W)$, and $\xi_j \in \mathfrak{O}$) has an inverse $\|v_{ij}\|$, with $v_{ij} \in \mathfrak{O}$. Set $D_i = \sum_j v_{ij} D'_j$. Then $D_i \in \mathfrak{D}(W)$ and $D_i\xi_j = \delta_{ij}$, from which it follows that the ξ_j are uniformizing coordinates of W.

In the proof of our next theorem, which characterizes uniformizing coordinates, we shall need a lemma concerning the function field of an arbitrary irreducible variety in S_n (over k) and in which *it is not assumed* that this function field is separable over k (as we later apply this lemma to the subvariety W of V).

LEMMA 1. *If an irreducible, s-dimensional variety W/k in S_n admits a field k_1 of p-constants and if $[k: k_1] = p^\rho$, then* $\dim \mathfrak{D}(k(W)/k_1) \leqq s + \rho$, *and any set of generators of $k(W)$ over k_1 contains a subset of σ elements, $z_1, \cdots, z_\sigma$, where $\sigma = \dim \mathfrak{D}(k(W)/k_1)$, such that $k(W)/k_1(z_1, \cdots, z_\sigma)$ is separable algebraic.*

Proof. By assumption, the ideal of W in $k[X_1, \cdots, X_n]$ has a basis $\{g_i(X_1, \cdots, X_n; u_1, \cdots, u_\rho);\ i = 1, 2, \cdots, N\}$ of polynomials with coefficients in $k^p[u_1, \cdots, u_\rho]$, where the u_i are p-parameters of W/k associated with the field k_1. Since W/k is simple on S_n, we know ([1] p. 39) that the mixed Jacobian matrix $J(g_1, \cdots, g_N; X_1, \cdots, X_n, u_1, \cdots, u_\rho)$ has rank $n - s$ on W (i.e., at a general point of W/k). If we denote by $Y_1, \cdots, Y_{n+\rho}$ the elements $X_1, \cdots, X_n, u_1, \cdots, u_\rho$ (in some order) and by $\xi_1, \cdots, \xi_{n+\rho}$ their $\mathfrak{d}(W)$-residues, then we may assume (say) that

$$\partial(g_1, \cdots, g_{n-s})/\partial(Y_{\rho+s+1}, \cdots, Y_{\rho+n}) \neq 0$$

at $(Y) = (\xi)$. This implies that $\xi_{\rho+s+1}, \cdots, \xi_{\rho+n}$ are separable algebraic over $k_1(\xi_1, \cdots, \xi_{\rho+s})$. As $k(W) = k_1(\xi_1, \cdots, \xi_{n+\rho})$, we have that $k(W)$ is separable algebraic over $k_1(\xi_1, \cdots, \xi_{\rho+s})$, and it is well-known ([2], Theorem 41, p. 127) that this implies that $\dim \mathfrak{D}(k(W)/k_1) \leqq s + \rho$.

Let $\{y_\mu\}$ be a set of generators of $k(W)/k_1$ and set $L = k(W)$. Then every derivation of L/k_1 is trivial on $k_1(L^p)$, and L is a purely inseparable extension of $k_1(L^p)$, with $[L: k_1(L^p)] = p^\sigma$, $\sigma = \dim \mathfrak{D}(L/k_1)$. We can extract from $\{y_\mu\}$ a set $\{y_1, \cdots, y_\sigma\}$ of elements such that

$$[k_1(L^p)(y_1, \cdots, y_\sigma) : k_1(L^p)] = p^\sigma,$$

and then we can construct σ linearly independent derivations $D_1, \cdots, D_\sigma$ of L/k_1 such that $D_i y_j = \delta_{ij}$ ([2] Corollary 5, p. 125). It is clear that any derivation of L over $k_1(y_1, \cdots, y_\sigma)$ vanishes on L. Therefore L is a separable

algebraic extension of $k_1(y_1, \cdots, y_\sigma)$. This completes the proof of the lemma.

THEOREM 1. *Let W be an irreducible subvariety of V/k and let $\{u_1, \cdots, u_\rho\}$ and k_1 be p-parameters and associated field of p-constants for V/k. If W is simple on V/k, then, in order that given elements $\xi_1, \cdots, \xi_{\rho+r}$ of $\mathfrak{O}$ be uniformizing coordinates of W, it is necessary and sufficient that the following two conditions be satisfied:*

(i) *$k(W)$ is a separable algebraic extension of $k_1(\bar{\xi}_1, \cdots, \bar{\xi}_{\rho+r})$, where $\bar{\xi}_i = \mathfrak{M}$-residue of ξ_i ($\mathfrak{O}$ is the local ring of W on V; $\mathfrak{M}$ is the maximal ideal of $\mathfrak{O}$);*

(ii) *$k_1[\xi_1, \cdots, \xi_{\rho+r}]$ contains a set of uniformizing parameters of W on V/k.*

Proof. Suppose that $\xi_1, \cdots, \xi_{\rho+r}$ are uniformizing coordinates of W. Let $\bar{D} \in \bar{\mathfrak{D}}(W)$ and assume that $\bar{D} = 0$ on $k_1(\bar{\xi}_1, \cdots, \bar{\xi}_{\rho+r})$. To establish (i) it will suffice to show that $\bar{D} = 0$. Let $\bar{D}^*$ be the local derivation of $\mathfrak{O}$ obtained by lifting $\bar{D}$ to $\mathfrak{D}^*(W)$. It will suffice to show that $\bar{D}^* = 0$. Let $D_1, \cdots, D_{\rho+r}$ be as in Definition 1 and let $D^*_1, \cdots, D^*_{\rho+r}$ be the corresponding elements of $\mathfrak{D}^*(W)$. Since $D^*_i \xi_j = \delta_{ij}$, the D^*_i are linearly independent over $k(W)$ and therefore form a basis of $\mathfrak{D}^*(W)$ over $k(W)$. However, $\bar{D}^* \xi_j = 0$ for $j = 1, \cdots, \rho + r$ and $D^*_i \xi_j = \delta_{ij}$ $(i, j = 1, \cdots, \rho + r)$. It follows immediately that $\bar{D}^* = 0$.

We first prove (ii) under the additional assumption that $u_1, \cdots, u_\rho$ and k_1 are p-parameters and associated field of p-constants not only for V/k but also for W/k.

From (i) it follows that $\dim \mathfrak{D}(k_1(\bar{\xi}_1, \cdots, \bar{\xi}_{\rho+r}) = \dim \bar{\mathfrak{D}}(W)$ (all derivations being assumed zero on k_1). Therefore, by Lemma 1,

$$\dim \mathfrak{D}(k_1(\bar{\xi}_1, \cdots, \bar{\xi}_{\rho+r})) \leqq \rho + s,$$

and thus, by known results on derivations of fields, it is possible to extract from $\{\bar{\xi}_1, \cdots, \bar{\xi}_{\rho+r}\}$ a subset of $\rho + s$ elements, say $\bar{\xi}_1, \cdots, \bar{\xi}_{\rho+s}$, such that $k_1(\bar{\xi}_1, \cdots, \bar{\xi}_{\rho+r})$ is a separable algebraic extension of $k_1(\bar{\xi}_1, \cdots, \bar{\xi}_{\rho+s})$ (compare with the proof of the second part of Lemma 1). Let $F_\lambda(\bar{\xi}_1, \cdots, \bar{\xi}_{\rho+s}, X)$ $(\lambda = 1, \cdots, r - s)$ be the minimal polynomial of $\bar{\xi}_{\rho+s+\lambda}$ over $k_1(\bar{\xi}_1, \cdots, \bar{\xi}_{\rho+s})$; F_λ may be assumed to be a polynomial in $\bar{\xi}_1, \cdots, \bar{\xi}_{\rho+s}$ (as well as in X). Let

$$\tau_\lambda = F_\lambda(\xi_1, \cdots, \xi_{\rho+s}, \xi_{\rho+s+\lambda}), \qquad \lambda = 1, \cdots, r - s. \tag{1}$$

Then $\tau_\lambda \in \mathfrak{M} \cap k_1[\xi_1, \cdots, \xi_{\rho+r}]$, and in order to prove (ii) it will be sufficient to show that if there is a relation of the form $\sum a_\lambda \tau_\lambda \in \mathfrak{M}^2$, $a_\lambda \in \mathfrak{O}$, then the a_λ must belong to $\mathfrak{M}$ (since the τ_λ will then be uniformizing parameters of W on V, i.e., regular parameters of $\mathfrak{O}$). Now we have $D(\sum a_\lambda \tau_\lambda) \in \mathfrak{M}$ for any D in $\mathfrak{D}(W)$, and, in particular, for $\mu = 1, \cdots, r-s$,

$$D_{\rho+s+\mu}(\sum_\lambda a_\lambda \tau_\lambda) = \sum_\lambda (D_{\rho+s+\mu} a_\lambda)\tau_\lambda + \sum_\lambda a_\lambda (D_{\rho+s+\mu}\tau_\lambda) \in \mathfrak{M}.$$

Hence $\sum_\lambda a_\lambda (D_{\rho+s+\mu}\tau_\lambda) \in \mathfrak{M}$. But, by (1), we have $D_{\rho+s+\mu}\tau_\lambda = 0$ if $\mu \neq \lambda$ and $D_{\rho+s+\lambda}\tau_\lambda \notin \mathfrak{M}$ (since $\bar{\xi}_{\rho+s+\lambda}$ is separable algebraic over $k_1(\bar{\xi}_1, \cdots, \bar{\xi}_{\rho+s})$ and therefore, $\frac{\partial F_\lambda}{\partial X}(\bar{\xi}_1, \cdots, \bar{\xi}_{\rho+s}, \bar{\xi}_{\rho+s+\lambda}) \neq 0$). Thus $a_\lambda (D_{\rho+s+\lambda}\tau_\lambda) \in \mathfrak{M}$ and $a_\lambda \in \mathfrak{M}$.

In the general case, we proceed as follows:

We fix in k_1 a finite set of p-independent elements $u_{\rho+1}, \cdots, u_\sigma$ such that the ideal $\mathfrak{J}(W)$ has a basis of polynomials with coefficients in $k^p(u_1, \cdots, u_\sigma)$. Then we fix a maximal subfield k_2 of k_1 such that

$$k^p < k_2, \quad k_1 = k_2(u_{\rho+1}, \cdots, u_\sigma) \quad \text{and} \quad [k_1 : k_2] = p^{\sigma-\rho}.$$

Then it is clear that $u_1, \cdots, u_\sigma$ are p-parameters, and k_2 is an associated field of p-constants, for both V/k *and* W/k. We now use the notations of the proof of Proposition 1. The coefficients of $f_1, \cdots, f_{n-r}$ are in $k^p(u_1, \cdots, u_\rho)$, and $J(f_1, \cdots, f_{n-r}; \eta_{\rho+r+1}, \cdots, \eta_{\rho+n}) \notin \mathfrak{M}$. It follows from that proof, as applied to $u_1, \cdots, u_\sigma$ and k_2, that $\eta_1, \cdots, \eta_{\rho+r}, u_{\rho+1}, \cdots, u_\sigma$ are uniformizing coordinates of W relative to k_2. Let $D'_1, \cdots, D'_{\rho+r}, D'_{\rho+r+1}, \cdots, D'_{\sigma+r}$ be the corresponding derivations of $\mathfrak{O}$ (over k_2). We have therefore

$$D'_i \eta_j = \delta_{ij}, \qquad (i, j = 1, 2, \cdots, \rho + r),$$

and

$$D'_i u_\nu = 0, \qquad (i = 1, 2, \cdots, \rho + r; \nu = \rho + 1, \cdots, \sigma). \tag{2}$$

The relations (2) show that $D'_i, \cdots, D'_{\rho+r}$ are zero on k_1. Since both $\eta_1, \cdots, \eta_{\rho+r}$ and $\xi_1, \cdots, \xi_{\rho+r}$ are uniformizing coördinates of W, over k_1, we have, by Proposition 2, that the determinant of the derivatives $D'_i \xi_j$ $(i, j = 1, 2, \cdots, \rho + r)$ does not belong $\mathfrak{M}$. In view of (2), and since $D'_{\rho+r+\alpha} u_{\rho+\beta} = \delta_{\alpha\beta}$, it follows that also the $(\sigma + r)$-row determinant of the derivatives

$$D'_q \xi_j, \; D'_q u_{\rho+\beta} \quad (q = 1, 2, \cdots, \sigma + r; \; j = 1, 2, \cdots, \rho + r; \; \beta = 1, 2, \cdots, \sigma - \rho)$$

is not in $\mathfrak{M}$. Therefore, by Proposition 2, $\xi_1, \cdots, \xi_{\rho+r}, u_{\rho+1}, \cdots, u_\sigma$ are uni-

formizing coördinates of W, over k_2. Hence, by the preceding special case of the proof, the ring $k_2[\xi_1, \cdots, \xi_{\rho+r}, u_{\rho+1}, \cdots, u_\sigma]$ contains a set of uniformizing parameters of W on V/k. Since $k_2[u_{\rho+1}, \cdots, u_\sigma] = k_1$, the proof of (ii) is now complete.

Conversely, assume that $\xi_1, \cdots, \xi_{\rho+r}$ satisfy conditions (i) and (ii). Fix a set $\{\eta_1, \cdots, \eta_{\rho+r}\}$ of uniformizing coordinates of W (Proposition 1) and let $D'_1, \cdots, D'_{\rho+r}$ be as in Proposition 2. It will suffice to show that $|D'_i\xi_j| \notin \mathfrak{M}$. Suppose this is false. Then there exist elements $a_1, \cdots, a_{\rho+r}$ of $\mathfrak{O}$, *not all in* $\mathfrak{M}$, such that $\sum_i a_i(D'_i\xi_j) \in \mathfrak{M}$ $(j = 1, \cdots, \rho + r)$. Let, say, $a_1 \notin \mathfrak{M}$ and set $D = \sum_i a_i D'_i$. Then $D \in \mathfrak{D}(W)$ and $D\xi_j \in \mathfrak{M}$, from which it follows that $D\mathfrak{M} \subset \mathfrak{M}$ (by (ii)). Hence $\bar{D} = \mathrm{Tr}_W D$ is defined (by $\bar{D}\bar{z} = \overline{Dz}$, $z \in \mathfrak{O}$) and is an element of $\bar{\mathfrak{D}}(W)$. Since $\bar{D}\xi_j = 0$, we have $\bar{D} = 0$ (by (ii)), in contradiction with $\bar{D}\eta_1 = \bar{a}_1 \neq 0$. This completes the proof of the theorem.

COROLLARY. *If W is as in the Theorem* 1, *then there exist uniformizing coordinates $\eta_1, \cdots, \eta_{\rho+r}$ of W such that $\eta_1, \cdots, \eta_{r-s}$ (say) are uniformizing parameters of W on V.*

Proof. Let $\xi_1, \cdots, \xi_{\rho+r}$ be uniformizing coordinates of W (Proposition 1) and suppose that $k(W)$ is separable algebraic over $k_1(\bar{\xi}_1, \cdots, \bar{\xi}_{\rho+s})$ (compare with the second paragraph of the proof of the theorem). Let $t_1, \cdots, t_{r-s}$ be uniformizing parameters of W on V and set

$$\eta_1 = t_1, \cdots, \eta_{r-s}, = t_{r-s}, \ \eta_{r-s+1} = \xi_1, \cdots, \eta_{\rho+s} = \xi_{\rho+s}.$$

It is immediate from the theorem that the η_i are uniformizing coordinates of W on V.

We remark that all the results established in this section continue to hold (and are well-known) also in the case of characteristic zero. In this case, we have $k = k^p$, $\rho = 0$, and $k_1 = k$.

2. The order of a differential at a prime divisor. From now on we shall assume that $r = 1$. Let $\mathfrak{p}$ be a prime divisor of K/k and let $v = v_\mathfrak{p}$ be the valuation of K/k defined by $\mathfrak{p}$. Since for function fields of one variable the notion of a prime divisor is equivalent with that of an irreducible, 0-dimensional subvariety W of a normal (hence non-singular) model V of K/k, we can speak of uniformizing coordinates of $\mathfrak{p}$, relative to a given set of p-parameters $u_1, \cdots, u_\rho$ and an associated field k_1 of p-constants.

Since the space $\mathfrak{D}(K)$ of derivations of K (over k_1) has dimension $\rho + 1$, we may speak of differentials of K, of degree $\rho + 1$, and for any $\rho + 2$ elements

$A, z_1, \cdots, z_{\rho+1}$ of K, we have a differential $A dz_1, \cdots, dz_{\rho+1}$ of K. Furthermore, if $\xi_1, \cdots, \xi_{\rho+1}$ are uniformizing coordinates of $\mathfrak{p}$, then every $(\rho+1)$-fold differential of K can be written in the form $A d\xi_1 \cdots d\xi_{\rho+1}$, $A \in K$ (as an immediate consequence of the fact that the derivations $D_1, \cdots, D_{\rho+1}$ of Definition 1 form a basis of $\mathfrak{D}(K)$).

DEFINITION 2. *If x is a separating transcendental of K/k, if $\xi_1, \cdots, \xi_{\rho+1}$ are uniformizing coordinates of $\mathfrak{p}$, and if $dx du_1 \cdots du_\rho = A d\xi_1 \cdots d\xi_{\rho+1}$ (where the u_i are the given p-parameters), then the order of dx at $\mathfrak{p}$ (in symbols: $v(dx)$) is the integer $v(A)$.*

PROPOSITION 3. *The order of dx at $\mathfrak{p}$ is well-defined, i.e., $v(dx)$ is independent of the choice of the uniformizing coordinates ξ_i, of the p-parameters u_ν, and of the field k_1 of p-constants.*

Proof.

(a) That our definition of the order of dx is independent of the choice of the $\xi_i(u_1, \cdots, u_\rho$ and k_1 being fixed) follows immediately from Proposition 2.

(b) We next remark that it will suffice to prove the proposition for one particular separating transcendental x of K/k, which *we shall take as one of the generators* $x_1, \cdots, x_n$ of K/k. For if x' is another separating transcendental of K/k and if we set $dx' du_1 \cdots du_\rho = A' d\xi_1 \cdots d\xi_{\rho+1}$, then $A' = A D_x x'$, where D_x is the (unique) derivation of K over k such that $D_x x = 1$.

(c) We now demonstrate that for fixed $u_1, \cdots, u_\rho$ our definition is independent of the field k_1 of p-constants (we recall that k_1 is any maximal subfield of k, containing k^p, such that $[k_1(u_1, \cdots, u_\rho) : k_1] = p^\rho$, and is therefore not uniquely determined). We may, by (a), choose as uniformizing coordinates a suitable subset $\eta_1, \cdots, \eta_{\rho+1}$ of the set $\{\eta_1, \cdots, \eta_{\rho+n}\}$ of $\mathfrak{d}(V)$-residues of the elements $X_1, \cdots, X_n, u_1, \cdots, u_\rho$ (arranged in some order). We use here the notations of the proof of Proposition 1. Let $K_0 = k^p(\eta_1, \cdots, \eta_{\rho+n})$. Then K_0 depends only on $u_1, \cdots, u_\rho$ (the generators $x_1, \cdots, x_n$ of K/k having been fixed; K_0 is in fact the function field of V over $k^p(u_1, \cdots, u_\rho)$). Let $D_1, \cdots, D_{\rho+1}$ be the derivations of K/k_1 such that $D_i \eta_j = \delta_{ij}$, $i, j = 1, \cdots, \rho+1$, and let $D^0{}_1, \cdots, D^0{}_{\rho+1}$ be their restrictions to K_0. The derivations $D^0{}_i$ of K_0 are independent of the choice of k_1 since they are uniquely determined by the conditions $D^0{}_i \eta_j = \delta_{ij}$ (compare with the proof of Proposition 1). Since $x \in K_0$ and since $A = \partial^0(x, u_1, \cdots, u_\rho)/\partial^0(\eta_1, \cdots, \eta_{\rho+1})$ (where the right hand side denotes the Jacobian determinant relative to the

derivations $D^0_1, \cdots, D^0_{\rho+1}$), our assertion that, for given p-parameters $u_1, \cdots, u_\rho$, the order of dx is independent of the choice of k_1 is proved.

(d) We next remark that if $w_1, \cdots, w_\rho$ are p-parameters such that $k^p(w_1, \cdots, w_\rho) = k^p(u_1, \cdots, u_\rho)$, then $dxdw_1 \cdots dw_\rho = cdxdu_1 \cdots du_\rho$, where $c \in k^p(u_1, \cdots, u_\rho)$, $c \neq 0$. For if $\delta_1, \cdots, \delta_\rho$ are the derivations of $k^p(u_1, \cdots, u_\rho)$ such that $\delta_i u_j = 0$ if $i \neq j$ and $\delta_i u_i = 1$, then c is equal to the Jacobian determinant $\partial(w_1, \cdots, w_\rho)/\partial(u_1, \cdots, u_\rho)$. It follows that $v(dx)$ is the same for two sets of p-parameters which generate the same field over k^p.

Let $\{u_1, \cdots, u_\rho\}$ and $\{w_1, \cdots, w_\tau\}$ be any two sets of p-parameters and let $[k^p(u_1, \cdots, u_\rho, w_1, \cdots, w_\tau) : k^p] = p^{\rho+\alpha} = p^{\tau+\beta} = p^\sigma$. We can enlarge the sets $\{u_1, \cdots, u_\rho\}$ and $\{w_1, \cdots, w_\tau\}$ to sets of p-parameters $\{u_1, \cdots, u_\sigma\}$ and $\{w_1, \cdots, w_\sigma\}$ such that $k^p(u_1, \cdots, u_\sigma) = k^p(w_1, \cdots, w_\sigma)$. Hence, in view of (d) the proof of our proposition will be complete if we show that the order of dx is unchanged when the set of p-parameters is enlarged.

Now let $\{u_1, \cdots, u_\rho\}$, $\{u_1, \cdots, u_\rho, u_{\rho+1}, \cdots, u_\sigma\}$, $\sigma > \rho$, be two sets of p-parameters one of which is contained in the other. Let k'_1 be a (fixed) field of p-constants associated with $\{u_1, \cdots, u_\sigma\}$. Then we can set, by (c), $k_1 = k'_1(u_{\rho+1}, \cdots, u_\sigma)$, so that $k'_1 \subset k_1$. We use the notations of the proof of Proposition 1. The coefficients of $f_1, \cdots, f_{n-r}$ are in $k^p(u_1, \cdots, u_\rho)$, and $J(f_1, \cdots, f_{n-r}; \eta_{\rho+r+1}, \cdots, \eta_{\rho+n}) \notin \mathfrak{M}$. It follows from that proof, applied to $u_1, \cdots, u_\sigma$ and k'_1, that $\eta_1, \cdots, \eta_{\rho+1}, u_{\rho+1}, \cdots, u_\sigma$ are uniformizing coordinates of $\mathfrak{p}$ relative to k'_1. If $D^0_{\rho+1}, \cdots, D^0_\sigma$ are the derivations of

$$k^p(\eta_1, \cdots, \eta_{\rho+n}, u_{\rho+1}, \cdots, u_\sigma)$$

such that $D^0_\alpha u_\beta = \delta_{\alpha\beta}$ $(\alpha, \beta = \rho+1, \cdots, \sigma)$ and if

$$dxdu_1 \cdots du_\sigma = Bd\eta_1 \cdots d\eta_{\rho+1} du_{\rho+1} \cdots du_\sigma,$$

then

$$\begin{aligned} B &= \partial^0(x, u_1, \cdots, u_\sigma)/\partial^0(\eta_1, \cdots \eta_{\rho+1}, u_{\rho+1}, \cdots, u_\sigma) \\ &= \partial^0(x, u_1, \cdots, u_\rho)/\partial^0(\eta_1, \cdots, \eta_{\rho+1}) = A, \end{aligned}$$

and our assertion is proved.

3. The main result. Let K/k be a separably generated function field in one variable and let K' be a finite separable algebraic extension of K. In all that follows, we shall tacity assume that a set of p-parameters $u_1, \cdots, u_\rho$ of both K/k and K'/k (and similarly a field k_1 of p-constants of both K/k and K'/k) have been fixed in advance.[2] Let $\mathfrak{p}$ be a prime divisor of K/k;

[2] I. e., p-parameters of a normal model of K/k and of a normal model of K'/k.

let $\mathfrak{p}'$ be a prime divisor of K'/k lying over $\mathfrak{p}$; and let x be a separating transcendental of K over k (and hence also of K' over k). We denote by dx^K (respectively $dx^{K'}$) the differential dx viewed as a differential of K (respectively of K'). We shall use the following notations:

$(dx^K)_\mathfrak{p}$ = local $\mathfrak{p}$-component of the divisor of dx^K, i.e., $v_\mathfrak{p}(dx^K)\cdot\mathfrak{p}$;

$(dx^{K'})_{\mathfrak{p}'}$ = local $\mathfrak{p}'$-component of the divisor of $dx^{K'}$, i.e., $v_{\mathfrak{p}'}(dx^{K'})\cdot\mathfrak{p}'$;

$(dx^K)_{\mathfrak{p}'}$ = local $\mathfrak{p}'$-component of the extension of the divisor (dx^K) to K', i.e., $e_{\mathfrak{p}'/\mathfrak{p}}v_\mathfrak{p}(dx^K)\mathfrak{p}$, where $e_{\mathfrak{p}'/\mathfrak{p}}$ is the (reduced) ramification index of $\mathfrak{p}'$ over $\mathfrak{p}$;

$\mathfrak{O} = \mathfrak{O}_\mathfrak{p}$ = valuation ring of $\mathfrak{p}$;

$\mathfrak{O}' = \mathfrak{O}_{\mathfrak{p}'}$ = valuation ring of $\mathfrak{p}'$;

$\bar{\mathfrak{O}}$ = integral closure of $\mathfrak{O}$ in K'.

The main object of this paper is the proof of the following (well-known) theorem:

THEOREM 2. $(dx^{K'})_{\mathfrak{p}'} = (dx^K)_{\mathfrak{p}'} + (\mathfrak{D}_{\bar{\mathfrak{O}}/\mathfrak{O}})_{\mathfrak{p}'}$, *where* $(\mathfrak{D}_{\bar{\mathfrak{O}}/\mathfrak{O}})_{\mathfrak{p}'}$ *is the local* $\mathfrak{p}'$*-component of the divisor of the different* $\mathfrak{D}_{\bar{\mathfrak{O}}/\mathfrak{O}}$ *of* $\bar{\mathfrak{O}}$ *over* $\mathfrak{O}$.

COROLLARY. *If* y *is a separating transcendental of* K/k *and if* $\mathfrak{r}$ *is a prime divisor of* K/k *such that* $v_\mathfrak{r}(y) \geqq 0$, *then* $v_\mathfrak{r}(dy) = v_\mathfrak{r}(\mathfrak{D}_{R/k[y]})$, *where* R *is the integral closure of* $k[y]$ *in* K.

In order to prove the corollary, we need only apply the theorem to the two fields $k(y)$ and K and note that the order of dy (viewed as a differential of $k(y)$) at any prime divisor of $k(y)$ which is not a pole of y, is equal to 0 (since $y, u_1, \cdots, u_\rho$ are obviously uniformizing coordinates of such a prime divisor, where $u_1, \cdots, u_\rho$ are the fixed p-parameters of K/k).

The proof of the theorem consists of a series of reductions, followed by a direct argument in two cases. In outline, the steps are:

(1) reduction to the case in which K'/K is a Galois extension; then, in the Galois case,

(2) reduction to the case in which K'/K is a cyclic extension of prime order;

(3) direct argument in the case of pure ramification

(4) direct argument in the case of purely inseparable residue field extension.

The remainder of the paper is divided into four sections which correspond to the steps in the proof.

Before we begin the proof, we remark that if $\mathfrak{p}'$ is unramified over $\mathfrak{p}$ (i.e., if $e_{\mathfrak{p}'/\mathfrak{p}}=1$ and $k(\mathfrak{p}')$ is a separable algebraic extension of $k(\mathfrak{p})$), then Theorem 2 is true. In fact, in that case, if $\xi_1,\cdots,\xi_{\rho+1}$ are uniformizing coordinates of $\mathfrak{p}$, then, by Theorem 1, $\xi_1,\cdots,\xi_{\rho+1}$ are uniformizing coordinates of $\mathfrak{p}'$. Since $e_{\mathfrak{p}'/\mathfrak{p}}=1$, we have, if $dxdu_1\cdots du_\rho = Ad\xi_1\cdots\xi_{\rho+1}$, that $v_{\mathfrak{p}'}(dx^K)=v_{\mathfrak{p}}(dx^K)=v_{\mathfrak{p}}(A)=v_{\mathfrak{p}'}(A)=v_{\mathfrak{p}'}(dx^{K'})$. However, ([2], p. 302, Theorem 28), $v_{\mathfrak{p}'}(\mathfrak{D}_{\bar{\mathfrak{O}}/\mathfrak{O}})=0$ and our assertion follows.

4. Reduction to the case in which K'/K is a Galois extension.

PROPOSITION 4. *Let K and K' be as in Theorem 2 and let K'' be a finite separable extension of K'. Then, if the theorem is true for any two of the extensions K'/K, K''/K, K''/K', the theorem is true for the third.*

Proof. Let $\mathfrak{O}=\mathfrak{O}_{\mathfrak{p}}$, $\mathfrak{O}'=\mathfrak{O}_{\mathfrak{p}'}$, $\mathfrak{O}''=\mathfrak{O}_{\mathfrak{p}''}$ (where $\mathfrak{p}''$ is a prime divisor of K''/k lying over $\mathfrak{p}'$) and let $^-$, $^=$ denote integral closure in K', K'' respectively. Assume that the theorem holds for the extensions K'/K and K''/K. Then, we have:

$$(dx^{K''})_{\mathfrak{p}''}=(dx^K)_{\mathfrak{p}''}+(\mathfrak{D}^{=}_{\bar{\mathfrak{O}}/\mathfrak{O}})_{\mathfrak{p}''},$$

$$(dx^{K'})_{\mathfrak{p}'}=(dx^K)_{\mathfrak{p}'}+(\mathfrak{D}_{\bar{\mathfrak{O}}/\mathfrak{O}})_{\mathfrak{p}'}.$$

However, ([2], p. 309, Theorem 31), $\mathfrak{D}^{=}_{\bar{\mathfrak{O}}/\mathfrak{O}}=\mathfrak{D}^{=}_{\bar{\mathfrak{O}}/\bar{\mathfrak{O}}}\cdot\bar{\bar{\mathfrak{O}}}\mathfrak{D}_{\bar{\mathfrak{O}}/\mathfrak{O}}$ and therefore we have:

$$\begin{aligned}(dx^{K''})_{\mathfrak{p}''}&=(dx^K)_{\mathfrak{p}''}+(\mathfrak{D}^{=}_{\bar{\mathfrak{O}}/\bar{\mathfrak{O}}})_{\mathfrak{p}''}+e_{\mathfrak{p}''/\mathfrak{p}'}\cdot(\mathfrak{D}_{\bar{\mathfrak{O}}/\mathfrak{O}})_{\mathfrak{p}'}\\&=e_{\mathfrak{p}''/\mathfrak{p}'}[(dx^K)_{\mathfrak{p}'}+(\mathfrak{D}_{\bar{\mathfrak{O}}/\mathfrak{O}})_{\mathfrak{p}'}]+(\mathfrak{D}^{=}_{\bar{\mathfrak{O}}/\bar{\mathfrak{O}}})_{\mathfrak{p}''}\\&=(dx^{K'})_{\mathfrak{p}''}+(\mathfrak{D}^{=}_{\bar{\mathfrak{O}}/\bar{\mathfrak{O}}})_{\mathfrak{p}''}.\end{aligned}$$

We now observe that $\mathfrak{O}'$ is a quotient ring of $\bar{\mathfrak{O}}$ with respect to a multiplicative set S, and therefore we have $\bar{\bar{\mathfrak{O}}}'=\bar{\bar{\mathfrak{O}}}_s$. It follows ([2], p. 299, Theorem 27) that $(\mathfrak{D}^{=}_{\bar{\mathfrak{O}}/\mathfrak{O}})_{\mathfrak{p}''}=(\mathfrak{D}^{=}_{\bar{\mathfrak{O}}'/\mathfrak{O}'})_{\mathfrak{p}''}$. On the other hand, $\mathfrak{O}''$ is a quotient ring of both $\bar{\bar{\mathfrak{O}}}$ and $\bar{\bar{\mathfrak{O}}}'$, with respect to the same multiplicative set ([2], p. 47, Theorem 18 is used implicitly here). Therefore, we have $(\mathfrak{D}^{=}_{\bar{\mathfrak{O}}'/\mathfrak{O}'})_{\mathfrak{p}''}=(\mathfrak{D}_{\mathfrak{O}''/\mathfrak{O}'})_{\mathfrak{p}''}$, and the assertion follows.

The cases where the theorem holds for some of the other pairs of extensions are handled in an entirely similar manner and are omitted.

PROPOSITION 5. *If Theorem 2 is valid for Galois extensions K'/K, it is valid for arbitrary extensions K'/K.*

Proof. Let K'' be the least normal extension of K containing K'. Then K''/K and K''/K' are Galois. Consequently, by Proposition 4 and our assumption, Theorem 2 holds for K'/K.

We shall, therefore, assume from now on that K' is a Galois extension of K.

5. Reduction to the case in which K'/K is a cyclic extension of prime order. Let K_T, K_Z be the splitting and inertia fields, respectively, of $\mathfrak{p}'$, and let $\mathfrak{p}_Z$, $\mathfrak{p}_T$ be the contractions of $\mathfrak{p}'$ to K_Z and K_T respectively. Then it is known that $\mathfrak{p}_Z$ is unramified over $\mathfrak{p}$ ([2], p. 291, proof of Theorem 23) and that $\mathfrak{p}_T$ is unramified over $\mathfrak{p}_Z$ ([2], p. 292, Theorem 24). Hence, by the remark made at the end of Section 3, our result holds for the extensions K_Z/K and K_T/K_Z. In view of Proposition 4, we need only prove the theorem for the extension K'/K_T, i.e., we may assume from now on that $K = K_T$. The assumption $K = K_T$ leads to the (well-known) conclusion that the Galois group of K'/K is solvable. It will then follow directly, from Proposition 4, that, if the theorem is true for cyclic Galois extension of prime order, then the theorem is true for K'/K.

We shall assume from now on that K'/K is cyclic, of prime order q. Then $q = ep^s$ ($e = e_{\mathfrak{p}'/\mathfrak{p}}$), and therefore, we have either $q = e$ and $k(\mathfrak{p}') = k(\mathfrak{p})$ [3] (case of pure ramification) or $e = 1$, $q = p$, and $k(\mathfrak{p}')$ is a purely inseparable extension of $k(\mathfrak{p})$, of degree p (case of purely inseparable residue field extension). Before we present the direct arguments for these two cases, we prove a lemma relating $\mathfrak{O}'$ and $\mathfrak{O}$ (recall that $\mathfrak{O}'$ is the valuation ring of $\mathfrak{p}'$ and that $\mathfrak{O}$ is the valuation ring of $\mathfrak{p}$).

LEMMA 2. *There exists an element λ in K' such that $\mathfrak{O}' = \mathfrak{O}[\lambda]$ in either of the two cases, and, in the case of pure ramification, λ may be taken to be a uniformizing parameter of $\mathfrak{p}'$.*

Proof. Let $\lambda \in \mathfrak{O}'$ be such that $k(\mathfrak{p}') = k(\mathfrak{p})(\bar{\lambda})$ if $k(\mathfrak{p}') \neq k(\mathfrak{p})$; if $k(\mathfrak{p}') = k(\mathfrak{p})$, then let λ be a uniformizing parameter of $\mathfrak{p}'$. Let $\tilde{\mathfrak{O}} = \mathfrak{O}[\lambda]$, $\tilde{\mathfrak{M}} = \mathfrak{M}' \cap \tilde{\mathfrak{O}}$, where $\mathfrak{M}'$ is the maximal ideal in $\mathfrak{O}'$. Since $\mathfrak{p}'$ is the only extension of $\mathfrak{p}$, $\mathfrak{O}'$ is integral over $\mathfrak{O}$ and therefore also over $\tilde{\mathfrak{O}}$. Furthermore, since $\tilde{\mathfrak{M}}$ is the only prime ideal of $\tilde{\mathfrak{O}}$, $\tilde{\mathfrak{O}}$ is a local domain. We observe that $\tilde{\mathfrak{O}}/\tilde{\mathfrak{M}} = \mathfrak{O}'/\mathfrak{M}' = k(\mathfrak{p})(\bar{\lambda})$ and also that $\tilde{\mathfrak{O}}$ contains a uniformizing parameter τ of $\mathfrak{p}'$, where $\tau = \lambda$ or $\tau =$ uniformizing parameter of $\mathfrak{p}$, depending on whether or not $q = e$. Since $\lambda \notin K$, K' is the quotient field of $\tilde{\mathfrak{O}}$. We assert that $\mathfrak{O}' = \tilde{\mathfrak{O}}$. Let $\xi \in \mathfrak{O}'$. Then $\xi = \xi_0 + \eta\tau$, where $\xi_0 \in \tilde{\mathfrak{O}}$, τ is our

[3] Since $K = K_T$.

uniformizing parameter of $\mathfrak{p}'$ in $\tilde{\mathfrak{O}}$, and $\eta \in \mathfrak{O}'$, since $\tilde{\mathfrak{O}}/\tilde{\mathfrak{M}} = \mathfrak{O}'/\mathfrak{M}'$. It follows that, for $j \geqq 0$, there exists a $\xi_j \in \tilde{\mathfrak{O}}$ such that $\xi - \xi_j \in \mathfrak{M}'^{j+1}$. Now K' is the quotient field of $\tilde{\mathfrak{O}}$ and so, $\xi = a/b$ with a, b in $\tilde{\mathfrak{O}}$. We then have $a - b\xi_j \in \mathfrak{M}'^{j+1} \cap \tilde{\mathfrak{O}}$, i.e., $a \in \mathfrak{O}b + \mathfrak{M}'^{j+1} \cap \tilde{\mathfrak{O}}$ for all $j \geqq 0$. It follows that $a \in \tilde{\mathfrak{O}}b$ and hence that $\xi \in \tilde{\mathfrak{O}}$. This completes the proof of the lemma.

We let $F(X) \in K[X]$ be the minimal polynomial of λ over K. Then, ([2], p. 303, Theorem 29), $\mathfrak{D}_{\mathfrak{O}'/\mathfrak{O}} = \mathfrak{O}'F'(\lambda)$, and therefore, we must show that we have $(dx^{K'})_{\mathfrak{p}'} = (dx^K)_{\mathfrak{p}'} + (F'(\lambda))_{\mathfrak{p}'}$ in both of our two cases (where λ is chosen as in Lemma 2).

6. Direct argument in the case of pure ramification. We assume now that $q = e$ and $k(\mathfrak{p}') = k(\mathfrak{p})$. Let $\xi_1, \cdots, \xi_{\rho+1}$ be uniformizing coordinates at $\mathfrak{p}$, where ξ_1 is a uniformizing parameter of $\mathfrak{p}$ (compare with the corollary to Theorem 1). Then, it follows from Theorem 1 that $\lambda, \xi_2, \cdots, \xi_{\rho+1}$ are uniformizing coordinates at $\mathfrak{p}'$. We set $\eta_1 = \lambda$, $\eta_j = \xi_j$ for $j = 2, \cdots, \rho+1$, and if D_i $(i = 1, \cdots, \rho+1)$ are derivations of K such that $D_i\xi_\mu = \delta_{i\mu}$ $(i, \mu = 1, \cdots, \rho+1)$, then we denote their extensions to K' by $\tilde{D}_i$. We let $E_1, \cdots, E_{\rho+1}$ be the derivations of K' such that $E_i\eta_j = \delta_{ij}$. We have $\tilde{D}_i\eta_1 = -F^{D_i}(\lambda)/F'(\lambda)$, $i = 1, \cdots, \rho+1$, where $F^{D_i}(\lambda)$ is obtained from $F(\lambda)$ by applying D_i to the coefficients of $F(\lambda)$. Hence we have $\tilde{D}_i = \sum_{\nu=1}^{\rho+1} a_{i\nu}E_\nu$, where the matrix $A = \| a_{i\nu} \|$ has the form:

$$\begin{pmatrix} -F^{D_1}(\lambda)/F'(\lambda) & 0 \cdots\cdots 0 \\ -F^{D_2}(\lambda)/F'(\lambda) & \\ \vdots & \boxed{\quad I_\rho \quad} \\ -F^{D_{\rho+1}}(\lambda)/F'(\lambda) & \end{pmatrix}.$$

Here I_ρ is the ρ by ρ identity matrix. It follows that, if $dxdu_1 \cdots du_\rho = Bd\xi_1 \cdots d\xi_{\rho+1}$, then

$$dxdu_1 \cdots du_\rho = B \cdot |A| \, d\eta_1 \cdots d\eta_{\rho+1} = -B(F^{D_1}(\lambda)/F'(\lambda))\, d\eta_1 \cdots d\eta_{\rho+1}.$$

Therefore it will suffice to show that $F^{D_1}(\lambda)$ is a unit in $\mathfrak{O}'$.

Let $F(\lambda) = \lambda^q + a_1\lambda^{q-1} + \cdots + a_q = 0$, with $a_i \in \mathfrak{O}$. Then we have $-a_q = \lambda^q + a_1\lambda^{q-1} + \cdots + a_{q-1}\lambda$, which implies that

$$\begin{aligned} qv_{\mathfrak{p}'}(a_q) &= v_{\mathfrak{p}'}(\lambda^{q-1} + \cdots + a_{q-1}) + 1 \\ &= 1 + \min\{q-1, q-2 + qv_{\mathfrak{p}}(a_1), \cdots, qv_{\mathfrak{p}}(a_{q-1})\}. \end{aligned}$$

Let $a_0 = 1$ and let a_{j_0} be the coefficient such that $a_{j_0} \notin \mathfrak{M}$ and $a_j \in \mathfrak{M}$ if $j > j_0$ (here necessarily $0 \leqq j_0 < q$). Then $qv_{\mathfrak{p}}(a_q) = q - j_0$. But $v_{\mathfrak{p}}(a_q) > 0$, and

therefore $j_0 = 0$ and $v_{\mathfrak{p}}(a_0) = 1$. Then $a_q = \epsilon\xi_1$, where ϵ is a unit in $\mathfrak{O}$, and we have $D_1 a_q = \xi_1 D_1\epsilon + \epsilon$, which implies that $D_1 a_q \notin \mathfrak{M}$. Since $F^{D_1}(\lambda) = (D_1 a_1)\lambda^{q-1} + \cdots + (D_1 a_q)$ and $D_1 a_i \in \mathfrak{O}$, while $v_{\mathfrak{p}'}(\lambda) = 1$, it follows that $v_{\mathfrak{p}'}(F^{D_1}(\lambda)) = 0$, and the proof for this case is now complete.

7. Direct argument in the case of purely inseparable residue field extension. We assume now that $q = p$ and that $k(\mathfrak{p}')$ is a purely inseparable extension of $k(\mathfrak{p})$. We let $\xi_1, \cdots, \xi_{\rho+1}$ be uniformizing coordinates of $\mathfrak{p}$, where we may assume (see Corollary to Theorem 1) that ξ_2 is a uniformizing parameter at $\mathfrak{p}$ (and hence also at $\mathfrak{p}'$). Now, $k(\mathfrak{p}')$ is (by our choice of λ) a separable algebraic extension of $k_1(\bar{\xi}_1, \bar{\xi}_3, \cdots, \bar{\xi}_{\rho+1}, \bar{\lambda})$ (where k_1 is our field of p-constants). We may assume that $u_1, \cdots, u_\rho$ and k_1 are p-parameters and associated field of p-constants not only for K/k and K'/k but also for $\mathfrak{p}'/k$. Under this assumption, we have, by Lemma 1, that $\dim \mathfrak{D}(k(\mathfrak{p}')/k_1) \leqq \rho$, and since $\bar{\lambda} \notin k(\mathfrak{p})$, we may assume, by the second part of Lemma 1, that $\bar{\xi}_1$ is separable algebraic over $k_1(\bar{\xi}_3, \cdots, \bar{\xi}_{\rho+1}, \bar{\lambda})$. It follows from Theorem 1 that if we let $\eta_1 = \lambda$, $\eta_j = \xi_j$, $j = 2, \cdots, \rho+1$, then the η_i are uniformizing coordinates of $\mathfrak{p}'$. As in Section 6, we let $\{D_i\}$ be the derivations of K/k_1 such that $D_i\xi_i = \delta_{ij}$; we let $\bar{D}_i$ denote the extension of D_i to K'; and we let $E_1, \cdots, E_{\rho+1}$ be the derivations of K' for which $E_i\eta_j = \delta_{ij}$. Arguing as in the previous section, we see that it will suffice to show that $F^{D_1}(\lambda)$ is a unit in $\mathfrak{O}'$.

Let $F(\lambda) = \lambda^p + a_1\lambda^{p-1} + \cdots + a_p = 0$, with $a_i \in \mathfrak{O}$. Since $k(\mathfrak{p})(\bar{\lambda})$ is purely inseparable over $k(\mathfrak{p})$, the coefficients $a_1, \cdots, a_{p-1}$ are in $\mathfrak{M}$. We observe that $D_1\mathfrak{M} \subset \mathfrak{M}$ and therefore that the trace of D_1 on $k(\mathfrak{p})$ (which we denote by $\bar{D}_1$) is defined.[4] If $D_1 a_p \in \mathfrak{M}$, then $\bar{D}_1$ can be extended to $k(\mathfrak{p})(\bar{\lambda})$, and we denote the extension by $\bar{D}'_1$. However, $\bar{D}'_1\bar{\eta}_j = 0$ for $j = 1, \cdots, \rho+1$ and $\bar{D}_1\bar{\xi}_1 = 1$ together contradict the fact that $\bar{\xi}_1$ is separable algebraic over $k_1(\bar{\eta}_1, \cdots, \bar{\eta}_{\rho+1})$. It follows that $D_1 a_p \notin \mathfrak{M}$. Since we have $F^{D_1}(\lambda) = (D_1 a_1)\lambda^{p-1} + \cdots + (D_1 a_p)$ and $D_1 a_i \in \mathfrak{M}$ if $i < p$, it follows at once that $v_{\mathfrak{p}'}(F^{D_1}(\lambda)) = 0$ and the proof is now complete also in this case.

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[4] It is given by: $\bar{D}_1 z = \overline{D_1 z}$, $z \in \mathfrak{O}$.

POLYHEDRA IN S^n WHOSE COMPLEMENTS HAVE UNIFORMLY ABELIAN LOCAL FUNDAMENTAL GROUPS.*[1]

By C. H. Edwards, Jr.

1. Introduction. If E and F are homeomorphic closed subsets of the n-sphere S^n, then a necessary condition that an arbitrary homeomorphism carrying E onto F will admit an extension to S^n is that the fundamental groups $\pi_1(S^n - E)$ and $\pi_1(S^n - F)$ of their complementary domains be isomorphic. O. G. Harrold [4] has investigated the problem of imposing local conditions on the complements of certain homeomorphic closed sets in S^n which will suffice to imply that the corresponding fundamental groups vanish. He shows in particular that if C is an i-cell in S^n with $S^n - C$ having uniformly abelian local fundamental groups, then $\pi_1(S^n - C) = 1$. A generalization of the proof of Harrold is here employed to extend this result to a certain class of contractible $(n-1)$-dimensional curvilinear polyhedra imbedded in S^n.

2. Notation and definitions. The notation and definitions used here are the same as those used by Harrold in [4]. In particular, the open set A in the space X will be said to have *uniformly abelian local fundamental groups* provided that for each point $p \in X$ and each neighborhood U of p in A, there is a neighborhood V of p such that $V \subset U$ and such that every closed path in V which is homotopic (in U) to the commutator of two paths in U is nullhomotopic in U.

3.

Theorem 1. *Suppose that C is a curved polyhedral complex imbedded in S^n, $n \leqq 5$, such that*

(a) $C = \bigcup_{i=1}^{k} D_i$, $i = 1, \cdots, k$, *being $(n-1)$-cell subcomplexes of C such that* $\operatorname{Int} D_j \cap \bigcup_{i \neq j} D_i = \square$, $j = 1, \cdots, k$.

(b) $D_j \cap \bigcup_{i=1}^{j-1} D_i$, $j = 2, \cdots, k$, *is an $(n-2)$-cell complex tamely imbedded in the $(n-2)$-sphere* $\operatorname{Bd} D_j$.

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If $S^n - C$ has uniformly abelian local fundamental groups, then $\pi_1(S^n - C) = 1$.

The dimensional restriction $n \leqq 5$ arises from the use of the following conjecture, which has been proved in the case $n \leqq 5$ by Guggenheim [3]:

Conjecture A. *If P^{n-2} is a polyhedral $(n-2)$-cell in S^{n-1}, then P^{n-2} is flat in S^{n-1}* (there exists a piecewise-linear isotopy of S^{n-1} carrying P^{n-2} onto a standard $(n-2)$-simplex).

If Conjecture A holds for all dimensions, then Theorem 1 holds for all dimensions. Observe that the tameness hypothesis in (b) is unnecessary for $n \leqq 5$.

4. Outline of proof. Generalizing the construction employed by Harrold in [4], a separable metric subspace $X = A \cup M$ of the Hilbert cube is defined, with A being ULC^1 and homeomorphic to $S^n - C$, and with M being a topological $(n-1)$-sphere constituting the boundary in X of A. M is then clearly LC^1 with $\pi_1(M) = 1$. It is shown also that $\pi_1(A, X) = 1$, so that a theorem of Eilenberg and Wilder [2, Theorem 8] applies directly to show that $\pi_1(A) = 1$, which implies that $\pi_1(S^n - C) = 1$, A and $S^n - C$ being homeomorphic.

5.

5.1. *Preliminaries.* By the order of a point $a \in C$ will be meant the positive integer $p^{n-1}(a, C) + 1$, where $p^{n-1}(a, C)$ is the $(n-1)$-dimensional Betti number of C around the point a [1, § 1]. Denote by $C^{(k)}$ the subset of all points of C of order k. It is clear that $C^{(2)} = \bigcup_{i=1}^{k} \operatorname{Int} D_i$ is an open (in C) dense subset of C.

Hence given any point $a \in \operatorname{Int} D_j$, $1 \leqq j \leqq k$, $\delta_a \geqq 0$ can be chosen sufficiently small that $S(a, \delta_a) - C = S(a, \delta_a) - D_j = U_a \cup V_a$, with U_a and V_a separated by D_j [1, § 2]. It may be assumed that the notation is chosen in such a manner that

$$U_j = \bigcup_{a \in \operatorname{Int} D_j} U_a \text{ and } V_j = \bigcup_{a \in \operatorname{Int} D_j} V_a$$

are disjoint open connected sets separated in $U_j \cup V_j \cup D_j$ by D_j. (This can be done as in [5, p. 280]). It will be said that a point $x \in S^n - C$ sufficiently near D_j lies on the positive or negative side of D_j according as $x \in U_j$ or $x \in V_j$ respectively.

Now let $\{p^j{}_m\}_{m=1}^{\infty}$ be a countable dense subset of $\operatorname{Int} D_j$. For each m

choose $\delta_m > \epsilon_m > 0$ such that $S(p^j{}_m, \delta_m) - C = U^m \cup V^m$ with $U^m \subset U_j$ and $V^m \subset V_j$ and such that any connected subset of $S^n - C$ joining a point in $U^m \cap S(p^j{}_m, \epsilon_m)$ with a point in $V^m \cap S(p^j{}_m, \epsilon_m)$ must have a diameter $> 2\epsilon_m$. For each m choose an arc $l_m = p^j{}_m q^j{}_m$, with $l_m - p^j{}_m \subset S(p^j{}_m, \epsilon_m) - C$, in such a way that each of the two subsequences

$$\{p^j{}_m \mid q^j{}_m \in U_j\} \text{ and } \{p^j{}_m \mid q^j{}_m \in V_j\}$$

is dense in D_j.

The countable dense subset $\bigcup_{j=1}^{k} \{p^j{}_m\}_{m=1}^{\infty}$ of C is then ordered in such a manner as to obtain a sequence $\{p_i\}_{i=1}^{\infty}$ such that $p_i \in \operatorname{Int} D_j$ if and only if $i \equiv j \pmod k$. That is, the subsequence of $\{p_i\}_{i=1}^{\infty}$ which is contained in $\operatorname{Int} D_j$ is precisely the subsequence $\{p_{j+mk}\}_{m=0}^{\infty}$.

5.2. *Definition of A.* Proceeding in this section as in 4.6 of [4], define a sequence $\{f_i\}_{i=1}^{\infty}$ of real-valued continuous functions on $S^n - C$ by $f_i(x) = \rho(x, q_i)$ for $x \in S^n - C$, where the "relative distance metric" ρ is defined on $S^n - C$ by $\rho(x, y) =$ g. l. b. of the diameters of all connected subsets of $S^n - C$ containing both x and y.

A homeomorphism h carrying $S^n - C$ into Hilbert space is defined as follows: If $(x_1, \cdots, x_n) = x \in S^n - C$, let $h(x) = \{x_1, \cdots, x_n, f_1(x), f_2(x), \cdots\}$. Denote by A the image set $h(S^n - C)$. Assume that a metric μ is defined in A as in 4.64 of [4].

5.3. Since $S^n - C$ by hypothesis has uniformly abelian local fundamental groups, the techniques used in 4.7 and 4.71 of [4] apply with slight modification to show that A is ULC^1.

5.4. *Definition of M.* Let $C_j = \bigcup_{i=1}^{j} D_i$ for each $j = 1, \cdots, k$ so that $C_1 = D_1$ and $C_k = C$. For each $j = 1, \cdots, k$ let the metric ρ_j be defined on $S^n - C_j$ by $\rho_j(x, y) =$ g. l. b. of the diameters of all connected subsets of $S^n - C_j$ containing both x and y, and define k sequences of real-valued functions $\{f_{ij}\}_{i=1}^{\infty}$, $j = 1, \cdots, k$, by $f_{ij}(x) = \rho_j(x, q_i)$.

Suppose now that D_s is an $(n-1)$-cell in C_j and let U_s and V_s be the corresponding open sets defined in 5.1, such that $U_s \cup V_s \cup \operatorname{Int} D_s$ is a neighborhood of $\operatorname{Int} D_s$ with U_s and V_s separated in $U_s \cup V_s \cup D_s$ by D_s. If $x \in \operatorname{Int} D_s$ and $i \equiv 1, 2, \cdots, j \pmod k$ then define $f_{ij}'(x) = \lim_{\bar{x} \to x} f_{ij}(x)$ for $\bar{x} \in U_s$ and $f_{ij}''(x) = \lim_{\bar{x} \to x} f_{ij}(x)$ for $x \in V_s$. If $i \equiv j+1, \cdots, k \pmod k$, then

define $f_{ij}'(x) = f_{ij}''(x) = 0$. Since two points close to x both lying either in U_s or V_s can be joined by a connected set of small diameter, it is clear that the functions f_{ij}' and f_{ij}'' are thereby uniquely defined.

If $x \in \operatorname{Int} D_s$, define

$$h_j'(x) = \{x_1, \cdots, x_n, f_{1j}'(x), f_{2j}'(x), \cdots\},$$
$$h_j''(x) = \{x_1, \cdots, x_n, f_{1j}''(x), f_{2j}''(x), \cdots\}.$$

If $y \in \operatorname{Bd} D_s$, define $h_j'(x) = \lim_{x \to y} h_j'(x)$ and $h_j''(y) = \lim_{x \to y} h_j''(x)$ for $x \in \operatorname{Int} D_s$. Then h_j' and h_j'', $j = 1, \cdots, k$, are mappings of D_s into a Hilbert cube (with the metric referred to in 5.2). If $x \in D_s \cap C^{(1)}$, then C_j does not separate sufficiently small neighborhoods of x, and it follows that $h_j'(x) = h_j''(x)$. But if $x \in D_s \cap C^{(k)}$, $k > 1$, then C_j does separate sufficiently small neighborhoods of x, and it follows that $h_j'(x) \neq h_j''(x)$.

5.41. An elementary continuity argument similar to that used in 4.8 of [4] suffices to show that h_j' and h_j'' are homeomorphisms on D_s, so that $h_j'(D_s)$ and $h_j''(D_s)$ are $(n-1)$-cells, $j = 1, \cdots, k$; $s = 1, \cdots, j$. The manner in which these homeomorphisms match on the boundaries of two intersecting cells D_r and D_s is described in the following section.

5.42. Suppose that z is a point of $C^{(k)}$ common to the boundaries of the $(n-1)$-cells D_r and D_s in C_j. If $\delta > 0$ is sufficiently small, then $S(z, \delta) - C_j$ consists of k mutually separated open sets. Suppose that W is one of these sets intersecting both U_r and U_s. Then it follows that $\lim h_j'(x)$ with $x \to z$, $x \in D_r$, is equal to $\lim h_j'(y)$ with $y \to z$, $y \in D_s$. To see this, notice that if $x \in \operatorname{Int} D_r$ and $y \in \operatorname{Int} D_s$ are sufficiently near z, and if $\bar{x} \in U_r$ and $\bar{y} \in U_s$ are sufficiently near x and y respectively, then $\bar{x}$ and $\bar{y}$ are both in W and can be joined in $S^n - C_j$ by a connected set of small diameter. Taking limits as $\bar{x} \to x$, $\bar{y} \to y$, and $x \to z \leftarrow y$ gives the desired result.

5.43. Now define $M_j = \bigcup_{i=1}^{j} [h_i'(D_i) \cup h_j''(D_i)]$, $j = 1, \cdots, k$. Clearly $M = M_k$ is the boundary in $X = A \cup M$ of A.

5.5. *Proof that M is an $(n-1)$-sphere.* It will now be shown that $M = M_k$ is an $(n-1)$-sphere. That M_1 is an $(n-1)$-sphere follows as in 4.8 of [4]. The induction proof will be completed by showing that the assumption that M_j is an $(n-1)$-sphere implies that M_{j+1} is an $(n-1)$-sphere, $j = 1, \cdots, k-1$.

For each $x \in L_{j+1} = C_j \cap D_{j+1}$, define $f_{ij}^*(x) = \lim f_{ij}(\bar{x})$ for $\bar{x} \to x$, $\bar{x} \in D_{j+1}$

if $i \equiv 1, \cdots, j \pmod{k}$ and $f_{ij}^*(x) = 0$ if $i \equiv j+1, \cdots, k \pmod{k}$. Then define

$$h_j^*(x) = \{x_1, \cdots, x_n, f_{1j}^*(x), f_{2j}^*(x), \cdots\}.$$

It follows from 5.41 that $h_j^*(L_{j+1})$ is homeomorphic with L_{j+1} and is hence an $(n-2)$-cell.

Now triangulate M_j in the following natural manner: Recalling that each D_i is a subcomplex of M_j, choose for $h_j'(D_j)$ and $h_j''(D_i)$ the subdivisions induced by the homeomorphisms h_j' and h_j'' and the original subdivision of D_i. Since $M_j = \bigcup_{i=1}^{j} [h_j'(D_i) \cup h_j''(D_i)]$, this suffices to triangulate M_j.

Since L_{j+1} is by hypothesis a subcomplex of C, it follows immediately that $h_j^*(L_{j+1})$ is a subcomplex of M_j, that is, that the $(n-2)$-cell $h_j^*(L_{j+1})$ is polyhedrally imbedded in the $(n-1)$-sphere M_j.

5.51. Now define a mapping ϕ of $\bigcup_{i=1}^{j} [h_{j+1}'(D_i) \cup h_{j+1}''(D_i)]$ onto M_j by setting $\phi(x) = h_j' h_{j+1}'^{-1}(x)$ for $x \in h_{j+1}'(D_i)$ and $\phi(x) = h_j'' h_{j+1}''^{-1}(x)$ for $x \in h_{j+1}''(D_i)$. Observe that ϕ is clearly 1-1 bicontinuous except at points of $h_{j+1}'(L_{j+1}) \cup h_{j+1}''(L_{j+1})$. Indeed ϕ carries each of the $(n-2)$-cells $h_{j+1}'(L_{j+1})$ and $h_{j+1}''(L_{j+1})$ homeomorphically onto the $(n-2)$-cell $h_{j+1}^*(L_{j+1})$, and carries the $(n-1)$-cells $h_{j+1}'(D_i)$ and $h_{j+1}''(D_i)$ homeomorphically onto the $(n-1)$-cells $h_j'(D_i)$ and $h_j''(D_i)$ respectively, $i = 1, \cdots, j$. It follows as in 5.42 that ϕ is single-valued.

Since $h_j^*(L_{j+1})$ is a polyhedral $(n-2)$-cell in the $(n-1)$-sphere M_j, there exists, by Conjecture A, a homeomorphism f of M_j onto itself carrying $h_j^*(L_{j+1})$ onto a standard $(n-2)$-simplex P.

Now let E be an $(n-1)$-simplex with base P and opposite vertex p. Let q be an interior point of P and let p' be an interior point of the interval qp. Then denote by E' the $(n-1)$-simplex with base P and opposite vertex p'. Let g be the mapping defined as follows:

a) $g(x) = x$ if $x \in \mathrm{Cl}(M_j - E)$;

b) If $x \in P$, denote by x' the point of intersection of the segment xp with $\mathrm{Bd}\, E' - P$. Then let g contract the segment xp linearly onto the segment $x'p$. Clearly g gives a homeomorphism of $M_j - P$ onto $M_j - E'$, but each point $a \in \mathrm{Int}\, P$ has two image points: $g'(a) = a \in P$ and $g''(a) \in \mathrm{Bd}\, E' - P$.

A homeomorphism ψ of $\bigcup_{i=1}^{j} [h_{j+1}'(D_i) \cup h_{j+1}''(D_i)]$ onto the closed $(n-1)$-cell $\mathrm{Cl}(M_j - E')$ is now defined. If $x \notin h_{j+1}'(L_{j+1}) \cup h_{j+1}''(L_{j+1})$, let $\psi(x) = gf\phi(x)$. If $x \in h_{j+1}'(L_{j+1})$ define $\psi(x) = g'f\phi(x)$ and if $x \in h_{j+1}''(L_{j+1})$

define $\psi(x) = g''f\phi(x)$. It is immediate that ψ is a homeomorphism, so that $\bigcup_{i=1}^{j} [h_{j+1}'(D_i) \cup h_{j+1}''(D_i)]$ is a closed $(n-1)$-cell.

5.52. It is shown that $h_{j+1}'(D_{j+1}) \cup h_{j+1}''(D_{j+1})$ is a closed $(n-1)$-cell. By 5.41, $h_{j+1}'(D_{j+1})$ and $h_{j+1}''(D_{j+1})$ are both closed $(n-1)$-cells. If $x \in \operatorname{Int} L_{j+1} \cup \operatorname{Int} D_{j+1}$, then $h_{j+1}'(x)$ and $h_{j+1}''(x)$ are distinct, while $h_{j+1}'(x) = h_{j+1}''(x)$ if $x \in \operatorname{Bd} D_{j+1} - \operatorname{Int} L_{j+1}$. Thus $h_{j+1}'(D_{j+1})$ and $h_{j+1}''(D_{j+1})$ are two $(n-1)$-cells meeting in $h_{j+1}(\operatorname{Bd} D_{j+1} - \operatorname{Int} L_{j+1})$. Since by hypothesis L_{j+1} is tamely imbedded in the $(n-2)$-sphere $\operatorname{Bd} D_{j+1}$, it follows that $\operatorname{Bd} D_{j+1} - \operatorname{Int} L_{j+1}$ is a closed $(n-2)$-cell tamely imbedded in $\operatorname{Bd} D_{j+1}$. Hence $h_{j+1}(\operatorname{Bd} D_{j+1} - \operatorname{Int} L_{j+1})$ is a closed $(n-2)$-cell tame in each of the boundaries of the two $(n-1)$-cells $h_{j+1}'(D_{j+1})$ and $h_{j+1}''(D_{j+1})$. It follows easily that $h_{j+1}'(D_{j+1}) \cup h_{j+1}''(D_{j+1})$ is a closed $(n-1)$-cell.

Consequently $M_{j+1} = \bigcup_{i=1}^{j+1} [h_{j+1}'(D_i) \cup h_{j+1}''(D_i)]$ is the union of the two closed $(n-1)$-cells $\bigcup_{i=1}^{j} [h_{j+1}'(D_i) \cup h_{j+1}''(D_i)]$ and $h_{j+1}'(D_{j+1}) \cup h_{j+1}''(D_{j+1})$ intersecting in their common boundary, the $(n-2)$-sphere $h_{j+1}'(L_{j+1}) \cup h_{j+1}''(L_{j+1})$. Hence M_{j+1} is an $(n-1)$-sphere, and it follows by induction that M is an $(n-1)$-sphere.

6. Since M is an $(n-1)$-sphere, M is LC^1 with $\pi_1(M) = 1$. Since the proof of 4.9 of [4] applies without alteration to show that $\pi_1(A, X) = 1$, Theorem 8 of [2] now implies that $\pi_1(A) = 1$. Since A was constructed homeomorphic to $S^n - C$, it follows that $\pi_1(S^n - C) = 1$. This completes the proof of Theorem 1.

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EXISTENCE OF UNIVERSAL CONNECTIONS.*

By M. S. Narasimhan and S. Ramanan.

1. Introduction. The purpose of this paper is to prove the existence of universal connections for principal bundles with a compact Lie group as structure group. We prove (Theorem 2) that given a compact Lie group G and a positive integer n, there exist a differentiable principal G-bundle E *and* a connection γ_0 on E such that any connection on a differentiable principal G-bundle P with base of dimension $\leqq n$ can be obtained as the inverse image of the connection γ_0 by a differentiable bundle homomorphism of P into E. As is well-known, the analogous problem for bundles without connections is treated in the topology of fibre bundles [1].

It is also known that the Stiefel bundles play the role of universal bundles for the unitary groups $U(k)$. One can define in a natural way a connection on every Stiefel bundle (§ 2). We prove that these connections themselves are universal for connections in $U(k)$-bundles. A precise formulation is found in Theorem 1.

In the unitary case the problem is first solved locally by explicit construction, the crucial step being the lemma in § 3. The local solutions are then pieced up with the help of a special type of covering by coordinate cells.

In the general case, the compact Lie group G is identified with a closed subgroup of a unitary group. Starting from a universal connection for this unitary group, a universal connection for G is constructed by generalizing the usual method of construction of an invariant connection in the principal bundle associated with a Lie group and a closed subgroup ([3], p. 45).

A theorem of A. Weil ([1], p. 57) asserts that the cohomology classes of the base of a principal G-bundle obtained by substitution of the curvature form of a connection on P in the invariant polynomials of G are independent of the connection. Our result seems to explain this invariance and in fact furnishes an alternate proof in the case of compact Lie groups.

For definitions of the notions related to connections in principal bundles we refer to [1] and [3]. We use connections and connection forms interchangeably. By "differentiable" we always mean "indefinitely differentiable."

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All manifolds, bundles, bundle homomorphisms and differential forms are assumed to be differentiable. Also all manifolds that occur are paracompact.

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2. Canonical connections in Stiefel bundles. Let $\boldsymbol{C}^N$ be the N-dimensional complex number space with O as origin. The Stiefel manifold $V(N,k)$ (with $N \geqq k$) of all unitary k-frames at O may then be identified with the left coset space $U(N)/I_k \times U(N-k)$ where I_k is the unit (k,k) matrix. To every frame $(v_1, \cdots, v_k)$ with $v_i = \sum_{j=1}^{N} b_{i,j} e_j$ where (e_j) is the canonical base in $\boldsymbol{C}^N$, we associate the (N,k)-matrix $A = (a_{i,j})$ with $a_{i,j} = b_{j,i}$. Since $(v_1, \cdots, v_k)$ is orthonormal, we have $\sum_{\alpha=1}^{N} b_{j,\alpha} b_{l,\alpha} = \delta_{j,l}$, i.e. A satisfies the condition $A^*A = I_k$ where A^* is the conjugate transpose of A. Thus $V(N,k)$ is identified with (N,k) matrices A satisfying $A^*A = I_k$. The action of $U(k)$ (resp. $U(N)$) on $V(N,k)$ to the right (resp. to the left) goes over under the above identification into multiplication of (N,k) matrices by unitary (k,k) matrices on the right (resp. by unitary (N,N) matrices on the left). Under the action of $U(k)$, $V(N,k)$ becomes a principal $U(k)$-bundle (known as the Stiefel bundle) with the Grassman manifold $G(N,k)$ of k-subspaces of $\boldsymbol{C}^N$ as base. $G(N,k)$ may again be identified with the left coset space $U(N)/U(k) \times U(N-k)$.

Let S be the (N,k) matrix-valued function on $V(N,k)$ which associates to each frame $(v_1, \cdots, v_k)$ the matrix A. Consider the (k,k) matrix-valued differential form S^*dS on $V(N,k)$. Since $S^*S = I_k$ for every frame, on differentiation we obtain $S^*dS + (dS^*)S = 0$, or again $S^*dS + (S^*dS)^* = 0$. Hence S^*dS has actually values in the Lie algebra $\mathfrak{u}(k)$ (which is the vector space of skew-Hermitian matrices) of $U(k)$.

PROPOSITION 1. *S^*dS is a connection form on the Stiefel bundle $V(N,k)$ which is invariant under the action of $U(N)$.*

In fact, if X_ξ is a tangent vector at $\xi \in V(N,k)$ and $s \in U(k)$, we shall denote by $X_\xi s$ the image of X_ξ under the differential of the map $\xi \to \xi s$ of $V(N,k)$. Then we have

$$\begin{aligned}(S^*dS)(X_\xi s) &= S^*(\xi s)(X_\xi s)(S) \\ &= s^* S^*(\xi)(X_\xi S) s \\ &= s^{-1}(S^*dS)(X_\xi) s.\end{aligned}$$

On the other hand, if $a \in \mathfrak{u}(k)$, we identify a with a tangent vector at e and denote by ξa the image of a under the differential of the map $s \to \xi s$ of G into $V(N,k)$. Then

$$\begin{aligned}(S^*dS)(\xi a) &= S(\xi)^*(\xi a)(S)\\ &= S(\xi)^*S(\xi)a\\ &= a,\end{aligned}$$

since $S(\xi)^*S(\xi) = I_k$. Hence S^*dS is a connection form on the Stiefel bundle. Moreover, if $t \in U(N)$, the left translation of the differential form by t yields $(tS)^*d(tS) = S^*t^*tdS = S^*dS$.

Remark. This connection will hereafter be referred to as the canonical connection and will be denoted by γ_0.

The horizontal subspace for this connection at the point $\xi_0 = \begin{pmatrix} I_k \\ 0 \end{pmatrix}$ of $V(N,k)$ may be described as follows: The tangent space at ξ_0 can be identified with (N,N) skew Hermitian matrices of the type $\begin{pmatrix} P & -Q^* \\ Q & 0 \end{pmatrix}$ where P is a (k,k) skew Hermitian matrix and Q is a rectangular $(N, N-k)$ matrix. The horizontal vectors at ξ_0 for the connection γ_0 are then given by matrices of the type $\begin{pmatrix} 0 & -Q^* \\ Q & 0 \end{pmatrix}$. This description together with the invariance under the action of $U(N)$ characterises the connection γ_0 completely.

Analogous statements are true for the real Stiefel manifold $W(N,k)$ and the corresponding $O(k)$-bundle. In particular $S'dS$ (where S' is the transpose of S) is a connection form on the Stiefel bundle, the corresponding horizontal subspace at ξ_0 being given by matrices of the type $\begin{pmatrix} 0 & -Q' \\ Q & 0 \end{pmatrix}$. This is easily seen to be the orthogonal complement of the vertical subspace at ξ_0 with respect to the killing form $\mathrm{tr}(\mathrm{ad}x\,\mathrm{ad}y)$ on $\mathfrak{o}(k)$, the Lie algebra of $O(k)$.

3. The local problem.

LEMMA. *Let U be an open subset of $\boldsymbol{R}^n$ and V a relatively compact open subset whose closure is contained in U. For every differential form α of degree 1 on U with values in $\mathfrak{u}(k)$ (the space of skew-Hermitian matrices), there exist differentiable functions $\phi_1, \cdots, \phi_{m'}$ in V with values in the space $\mathfrak{M}_k(\boldsymbol{C})$ of (k,k) complex matrices such that*

i) $$\sum_{j=1}^{m'} \phi_j^*\phi_j = I_k, \quad \textit{and}$$

ii) $$\sum_{j=1}^{m'} \phi_j^* d\phi_j = \alpha,$$

where $m' = (2n+1)k^2$.

Proof. Let $f_1, \cdots, f_{k^2}$ be a set of positive definite matrices which form a base for the complex Hermitian matrices over the reals, such that $\| f_r \| = 1$ for every r ($\| f \|$ being the norm as a linear transformation). Since α has values in $\mathfrak{u}(k)$, we may write α/i in U as $\sum_{s=1}^{n} \sum_{r=1}^{k^2} \lambda_{r,s} f_r dx_s$, where $\lambda_{r,s}$ are real-valued functions and x_s the coordinate functions in $\boldsymbol{R}^n$. If $a_{r,s} = \sup_V | \lambda_{r,s} |$, we have $\lambda_{r,s} = \mu_{r,s} - \nu_{r,s}$ where

$$\mu_{r,s} = \tfrac{1}{2}\{\lambda_{r,s} + a_{r,s} + 1\} \quad \text{and}$$

$$\nu_{r,s} = \tfrac{1}{2}\{a_{r,s} - \lambda_{r,s} + 1\}$$

are both strictly positive differentiable functions. Hence we may write $\mu_{r,s} = p_{r,s}^2$ and $\nu_{r,s} = q_{r,s}^2$ where $p_{r,s}$ and $q_{r,s}$ are positive differentiable functions. Clearly one may assume that $\sum_{s=1}^{n} (\mu_{r,s} + \nu_{r,s})$ is bounded by $1/2k^2$ on V for every r, by altering the coordinate functions x_s by a constant multiple, if necessary. The matrix valued function $1/k^2 I_k - \{\sum_{s=1}^{n} (\mu_{r,s} + \nu_{r,s})\} f_r$ is then positive. For,

$$\| \sum_{s=1}^{n} (\mu_{r,s} + \nu_{r,s}) f_r \| \leqq 1/2k^2 \| f_r \| < 1/k^2.$$

Let g_r be the (unique) positive square-root of the positive matrix f_r and h_r the differentiable positive matrix-valued function satisfying

$$h_r^2(x) = 1/k^2 I_k - \{ \sum_{s=1}^{n} (\mu_{r,s}(x) + \nu_{r,s}(x)) \} f_r.$$

We now define $\mathfrak{M}_k(\boldsymbol{C})$-valued functions $\phi_j (1 \leqq j \leqq (2n+1)k^2)$ as follows.

For $1 \leqq j \leqq nk^2$, ϕ_j shall be the nk^2 functions $p_{r,s} e^{ix_s} \cdot g_r$ arranged in some order.

For $nk^2 + 1 \leqq j \leqq 2nk^2$, ϕ_j shall be the nk^2 functions $q_{r,s} e^{-ix_s} \cdot g_r$ arranged in some order.

For $2nk^2 + 1 \leqq j \leqq (2n+1)k^2$, ϕ_j shall be the functions h_r in some order.

We have to verify that the ϕ_j thus defined fulfil the conditions i) and ii) of the lemma. In fact,

$$\sum_{j=1}^{m'} \phi_j^* \phi_j = \sum_{r,s} p_{r,s}^2 g_r^2 + \sum_{r,s} q_{r,s}^2 g_r^2 + \sum_r h_r^2$$
$$= \sum_{r,s} \mu_{r,s} f_r + \sum_{r,s} \nu_{r,s} f_r + I_k - \sum_{r,s} (\mu_{r,s} + \nu_{r,s}) f_r$$
$$= I_k.$$

On the other hand,

$$\sum_{j=1}^{m'} \phi_j^* d\phi_j = \sum_{r,s} p_{r,s} e^{-ix_s} \{p_{r,s} i e^{ix_s} (dx_s) + (dp_{r,s}) e^{ix_s}\} g_r^2$$
$$+ \sum_{r,s} q_{r,s} e^{ix_s} \{-q_{r,s} \cdot i e^{-ix_s} (dx_s) + (dq_{r,s}) e^{-ix_s}\} g_r^2$$
$$+ \sum_r h_r dhr$$
$$= \sum_{r,s} i \cdot p_{r,s}^2 g_r^2 dx_s + \sum_{r,s} p_{r,s} dp_{r,s} g_r^2$$
$$+ \sum_{r,s} i(-q_{r,s}^2) g_r^2 dx_s + \sum_{r,s} q_{r,s} dq_{r,s} g_r^2 + \sum_r h_r dh_r$$
$$= \sum_{r,s} i(\mu_{r,s} - \nu_{r,s}) f_r dx_s + \tfrac{1}{2} \sum_{r,s} d(p_{r,s}^2 + q_{r,s}) f_r + \sum_r h_r dh_r$$
$$= \alpha + \tfrac{1}{2} \sum_{r,s} d(\mu_{r,s} + \nu_{r,s}) f_r + \sum_r h_r dh_r.$$

But since for any $x, y \in V$, $h_r^2(x)$ and $h_r^2(y)$ commute, their positive square roots $h_r(x)$ and $h_r(y)$ also commute. It readily follows that $h_r dh_r = dh_r \cdot h_r$. Hence $\frac{1}{2} d(h_r^2) = h_r dh_r$. Therefore, finally we have

$$\sum_{j=1}^{m'} \phi_j^* d\phi_j = \alpha + \tfrac{1}{2} d\{\sum_{r,s} (\mu_{r,s} + \nu_{r,s}) f_r + \sum_r h_r^2)\}$$
$$= \alpha,$$

since $\sum_{r,s} (\mu_{r,s} + \nu_{r,s}) f_r + \sum_r h_r^2 = I_k$, and the lemma is completely proved.

The problem is solved locally by the following

PROPOSITION 2. Let P be a principal $U(k)$-bundle over a manifold X of dimension $\leqq n$ and γ a connection form on P. For every relatively compact open subset W of X with $\bar{W}$ contained in a coordinate neighborhood U of X over which P is trivial, there exists a differentiable bundle map Φ of $p^{-1}(W)$ into $V(m'', k)$ such that the inverse image of the canonical connection γ_0 by Φ is γ, where $m'' = (2n+1)k^3$ and p is the projection $P \to X$.

Proof. Let σ be a section of P over U and α the inverse image of γ by σ.

By the lemma, we can find differentiable $\mathfrak{M}_k(\boldsymbol{C})$-valued functions $\phi_1, \cdots, \phi_{m'}$ in W such that

i) $\sum_{j=1}^{m'} \phi_j^* \phi_j = I_k$, and

ii) $\sum_{j=1}^{m'} \phi_j^* d\phi_j = \alpha$, where $m' = (2n+1)k^2$.

Define a map Φ of P over W into the space of (m'', k)-matrices by setting for $\xi \in P$, $\Phi(\xi) = \begin{pmatrix} \phi_1(p\xi) \\ \vdots \\ \phi_{m'}(p\xi) \end{pmatrix} \cdot s$ where $s \in U(k)$ is determined by $\xi = \sigma(p\xi)s$. Φ is easily seen to be a bundle homomorphism. We then have

$$\begin{aligned} \Phi^*\Phi(\xi) &= s^*\left(\sum_{j=1}^{m'} (\phi_i^* \phi_i)(p\xi)\right) s \\ &= s^* s, \text{ by (i)} \\ &= I_k, \text{ since } s \text{ is unitary.} \end{aligned}$$

Hence Φ maps $P|p^{-1}(W)$ actually into $V(m'', k)$. On the other hand, it is obvious that the inverse image by Φ of $\gamma_0 = S^* dS$ is given by $\Phi^* d\Phi$. But the inverse image by σ of $\Phi^* d\Phi$ is $(\Phi \circ \sigma)^* d(\Phi \circ \sigma) = \sum_{i=1}^{m'} \phi_i^* d\phi_i = \alpha$ by construction. Now γ and $\Phi^* d\Phi$ are two connections on $P|p^{-1}(W)$ such that their inverse image by the section σ are the same. Hence $\gamma = \Phi^* d\Phi$ on $p^{-1}(W)$.

4. Universal connection for the unitary group.

THEOREM 1. *Let P be a principal $U(k)$-bundle over a manifold X of dimension $\leqq n$ and γ any connection form on P. Then there exists a differentiable bundle homomorphism Φ of P into the Stiefel bundle $V(m, k)$ such that γ is the inverse image by Φ of the canonical connection γ_0 on $V(m, k)$, where $m = (n+1)(2n+1)k^3$.*

Proof. We can find a covering of X by relatively compact open sets $\{V_i\}$ such that i) each $\bar{V}_i$ is contained in a coordinate cell, and ii) the V_i's can be divided into $(n+1)$ classes $\mathcal{B}_j$ in such a way that no two V_i's of the same class intersect ([2], p. 61). Let $\{W_i\}$ be a shrinking of this covering, i.e., an open covering $\{W_i\}$ such that $\bar{W}_i \subset V_i$. Let D_j $(j = 1, \cdots, n+1)$ be the union of the open sets $p^{-1}(W_i)$ where $\bar{W}_i \subset V_i$ with V_i belonging to $\mathcal{B}_j$.

The bundle is trivial over the coordinate cells and hence, by Proposition 2, one can find differentiable bundle homomorphisms Φ_i on $p^{-1}(V_i)$ into

$V((2n+1)k^3, k)$ inducing the connection γ on $p^{-1}(V_i)$. Corresponding to each D_j there exists a $\{(2n+1)k^3, k\}$-matrix-valued differentiable function Ψ on P such that Ψ coincides with Φ_i on $p^{-1}(W_i)$ for V_i in $\mathfrak{B}_j$. Let then $\Psi_1, \cdots, \Psi_{n+1}$ be the $(n+1)$ functions thus constructed. Consider a differentiable partition of unity with respect to the covering $\{D_j\}$ consisting of non-negative differentiable functions ζ_i invariant under the action of the group $U(k)$ such that the support of $\zeta_i \subset D_i$ and $\sum \zeta_i^2 = 1$.

Consider now the map Φ on P defined by

$$\Phi(\xi) = \begin{pmatrix} \zeta_1(\xi)\Psi_1(\xi) \\ \vdots \\ \zeta_{n+1}(\xi)\Psi_{n+1}(\xi) \end{pmatrix} \text{ for every } \xi \in P.$$

We have to prove that Φ is bundle map of P into $V(m,k)$ such that $\Phi^* d\Phi = \alpha$. But

$$\begin{aligned} \Phi^*\Phi(\xi) &= \sum_{i=1}^{n+1} \zeta_i(\xi)^2 \Psi_i^*(\xi)\Psi_i(\xi) \\ &= \sum \zeta_i(\xi)^2 \Psi_i^*(\xi)\Psi_i(\xi), \end{aligned}$$

the summation being over those i's for which $\xi \in D_i$. But on D_i, $\Psi_i^*\Psi_i = I$ and we have $(\Psi_i^*\Psi_i) = I$ for every i over which the summation extends. Hence $\Phi^*\Phi(\xi) = \sum \zeta_i(\xi)^2 I = I$, since $\sum \zeta_i^2(\xi) = 1$. Moreover

$$\begin{aligned} \Phi(\xi s) &= \begin{pmatrix} \zeta_1(\xi s)\Psi_1(\xi s) \\ \vdots \\ \zeta_{n+1}(\xi s)\Psi_{n+1}(\xi s) \end{pmatrix} = \begin{pmatrix} \zeta_1(\xi)\Psi_1(\xi)s \\ \vdots \\ \zeta_{n+1}(\xi)\Psi_{n+1}(\xi)s \end{pmatrix} \\ &= \begin{pmatrix} \zeta_1(\xi)\Psi_1(\xi) \\ \vdots \\ \zeta_{n+1}(\xi)\Psi_{n+1}(\xi) \end{pmatrix} s \\ &= \Phi(\xi) \cdot s \end{aligned}$$

for every $\xi \in P$ and $s \in U(k)$.

Finally,

$$\begin{aligned} \Phi^* d\Phi &= \sum_{i=1}^{n+1} \zeta_i \Psi_i^* (d\zeta_i \Psi_i + \zeta_i d\Psi_i) \\ &= \sum_{i=1}^{n+1} \Psi_i^* \Psi_i \zeta_i d\zeta_i + \sum_{i=1}^{n+1} \zeta_i^2 \Psi_i^* d\Psi_i. \end{aligned}$$

As before, for a $\xi \in P$, the summation needs to be taken only over those i's

for which $\xi \in D_i$. In every such D_i, however, $\Psi_i^* \Psi_i = I$ and $\Psi_i^* d\Psi_i = \alpha$. Hence $\Phi^* d\Phi = \sum \zeta_i d\zeta_i I + (\sum \zeta_i^2)\alpha = \alpha$ since $\sum \zeta_i d\zeta_i = \frac{1}{2} d(\sum \zeta_i^2) = 0$.

5. Universal connections for compact Lie groups. Let G_2 be a closed subgroup of a Lie group G_1, and $\mathfrak{g}_2$ and $\mathfrak{g}_1$ their Lie algebras. The group G_2 acts on $\mathfrak{g}_1$ by the adjoint operations and $\mathfrak{g}_2$ is invariant under this representation. Suppose $\mathfrak{m}$ is a subspace of $\mathfrak{g}_1$ invariant under the action of G_2 which is supplementary to $\mathfrak{g}_2$. (Such a space $\mathfrak{m}$ exists if G_2 is compact or semi-simple.) Let P be a principal bundle with group G_1 and ω_1 a connection on P. P is fibred by G_2 into a principal bundle with group G_2.

The direct sum decomposition $\mathfrak{g}_2 \oplus \mathfrak{m}$ of $\mathfrak{g}_1$ gives rise to a projection π of $\mathfrak{g}_1$ onto $\mathfrak{g}_2$ which commutes with the action of G_2 (i.e. $\pi \circ ads = ads \circ \pi$ for every $s \in G_2$). We define a differential form ω_2 on P by setting $\omega_2 = \pi \circ \omega_1$. It is easy to see that ω_2 is a connection on P for the fibration by G_2. In fact, (with the notations of § 2),

$$\begin{aligned} \omega_2(X_\xi s) &= (\pi \cdot \omega_1)(X_\xi s) \\ &= \pi \cdot \omega_1(X_\xi s) \\ &= \pi \cdot \mathrm{ad} s\, \omega_1(X_\xi) \\ &= \mathrm{ad} s \cdot \pi\omega_1(X_\xi) \\ &= s^{-1}\omega_2(X_\xi)s \end{aligned}$$

for every vector X_ξ at $\xi \in P$ and $s \in G_2$.

On the other hand, for every $\xi \in P$ and $a \in \mathfrak{g}_2$, we have

$$\begin{aligned} \omega_2(\xi a) &= (\pi \cdot \omega_1)(\xi a) \\ &= \pi \cdot \omega_1(\xi a) \\ &= \pi(a) \\ &= a, \text{ since } \pi \text{ is a projection.} \end{aligned}$$

THEOREM 2. *Let G be a compact Lie group and n a positive integer. There exist a principal G-bundle B and a connection form γ_1 on B such that for every principal G-bundle P with base of dimension $\leqq n$ and any connection form γ on P, one can find a bundle homomorphism f of P into B such that the inverse image of γ_1 by f is γ.*

Proof. G can be identified with a closed subgroup of a unitary group $U(k)$. Let γ_0 be a universal connection for $U(k)$ (for the dimension n) on a principal $U(k)$-bundle B, whose existence has been proved in Theorem 1. G acts on B and makes of it a principal G-bundle. One can define a

connection γ_1 on this bundle by setting $\gamma_1 = \pi \circ \gamma_0$ where π is a projection of $\mathfrak{u}(k)$ onto the Lie algebra $\mathfrak{g}$ of G, as explained above. For any principal G-bundle P, with base of dimension $\leqq n$, let P' be the corresponding principal $U(k)$-bundle obtained by enlarging the group G. Then there is a natural inclusion $i: P \to P'$ such that $i(\xi s) = i(\xi)(s)$ for $\xi \in P$ and $s \in G$.

Moreover, if γ is a connection form on P, one can define a natural connection γ' on P' such that the inverse image of γ' by i is γ ([3], p. 35). Let Φ be a bundle map of P into B such that the inverse image of γ_0 by Φ is γ'. We define a bundle map f of P into B fibred by G by setting $f = \Phi \cdot i$. The inverse image of γ_0 by $f = \Phi \cdot i$ is obviously γ. But since γ has values in $\mathfrak{g}$ and π is identity on $\mathfrak{g}$, we have $\pi \cdot \gamma = \gamma$ and hence the inverse image of γ_1 by f is γ.

6. Remarks.

i) We show how A. Weil's theorem on connections can be deduced from our results, at least when G is compact. Let ω_1, ω_2 be two connections on a principal G-bundle P with base X of dimension $\leqq n-1$. Consider the bundle $P \times I' \to X \times I'$ where I' is the open interval $(-\epsilon, 1+\epsilon)$, $\epsilon > 0$. Let α_1, α_2 be inverse images of ω_1, ω_2 respectively under the projection $P \times I' \to P$. The differential form $\alpha = t\alpha_1 + (1-t)\alpha_2$ where t is the projection $P \times I' \to I'$, is easily seen to be a connection on $P \times I'$. Let B be a principal G-bundle over a manifold M and γ_1 a universal connection on B for dimension $\leqq n$. It follows that there exists a differentiable family F_t of differentiable bundle maps of P into B such that the inverse image of γ_1 by F_t is $t\omega_1 + (1-t)\omega_2$. If f_t are the corresponding maps of X into M, then f_0 and f_1 are obviously homotopic. On the other hand, if K_1 and K_2 are the curvature forms of ω_1, ω_2 respectively, the 'substitution' of K_1, K_2 in each polynomial over $\mathfrak{g}$ invariant under the adjoint representation of G yields closed differential forms β_1, β_2 on X. Then β_1 and β_2 are the inverse images under f_0 and f_1 of the form on M obtained by substituting the curvature form K of the universal connection in the same polynomial. Since f_0 and f_1 are homotopic, it follows that β_1 and β_2 define the same cohmology class on the base, a characteristic class of the bundle.

ii) Our method gives a universal connection for the orthogonal group $O(k)$ in particular. But the connection was defined in the complex Stiefel manifold fibred by $O(k)$ instead of the more usual real Stiefel bundle. We have already remarked (§ 2) that if the points of the real Stiefel manifold $W(N,k)$ are represented by (N,k)-matrices A satisfying $A'A = I_k$ (A' being

the transpose of A), a connection can be defined in a canonical way on the real Stiefel bundle with the corresponding connection form $A'dA$. On the other hand, the complex Stiefel manifold $V(N,k)$ may be imbedded into the real Stiefel manifold $W(2N,k)$ by associating to each (N,k) matrix A, the $(2N,k)$ real matrix $\tilde{A} = \begin{pmatrix} \mathrm{Rl}\ A \\ \mathrm{Im}\ A \end{pmatrix}$ where $\mathrm{Rl}\,A$, $\mathrm{Im}\,A$ are the real and imaginary parts of A. It is easy to see that if $A^*A = I_k$, we have $\tilde{A}'\tilde{A} = I_k$. Moreover, for every $s \in O(k) \subset U(k)$

$$(\widetilde{As}) = \begin{pmatrix} \mathrm{Rl}\,(As) \\ \mathrm{Im}\,(As) \end{pmatrix} = \begin{pmatrix} (\mathrm{Rl}\,A)s \\ (\mathrm{Im}\,A)s \end{pmatrix} = \tilde{A} \cdot s.$$

Hence the map $A \to \tilde{A}$ is a bundle map of the complex Stiefel manifold $V(N,k)$ fibred by $O(k)$, into the real Stiefel bundle. The connection form on $V(N,k)$ induced by this map is $\tilde{A}'d\tilde{A}$, but this is the same as the real part of A^*dA. If we take for the projection π of § 4, the map of $\mathfrak{u}(k)$ onto the Lie algebra $\mathfrak{o}(k)$ of $O(k)$ defined as the assignment of the real part to each skew Hermitian matrix, then the corresponding connection γ_1 is the same as the real part of A^*dA. In other words, the canonical connection in the real Stiefel bundle is universal for $O(k)$-bundles.

TATA INSTITUTE OF FUNDAMENTAL RESEARCH, BOMBAY.

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ON THE PREPARATION OF MANUSCRIPTS.

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Manuscripts not complying with the standards usually have to be returned to the authors for typographic explanation or revisions and the resulting delay often necessitates the deferment of the publication of the paper to a later issue of the Journal.

Horizontal fraction signs should be avoided. Instead of them use either solidus signs / or negative exponents.

Neither a solidus nor a negative exponent is needed in the symbols $\frac{1}{2}$, $\frac{1}{\pi}$, $\frac{1}{2\pi}$, $\frac{1}{2\pi i}$, which are available in regular size type.

Binomial coefficients should be denoted by C_k^n and not by parentheses. Correspondingly, for symbols of the type of a quadratic residue character the use of some non-vertical arrangement is usually imperative.

Use the exponent $\frac{1}{2}$ instead of the square root sign.

Replace $e^{(\ )}$ by exp( ) if the expression in the parenthesis is complicated.

By an appropriate choice of notations, avoid unnecessary displays.

Simple formulae, such as $A + iB = \frac{1}{2}C^*$ or $s_n = a_1 + \cdots + a_n$, should not be displayed (unless they need a formula number).

Use $'$ or $d\ /dx$, possibly D, but preferably not a dot, in order to denote ordinary differentiation and, as far as possible, a subscript in order to denote partial differentiation (when the symbol ∂ cannot be avoided, it should be used as $\partial\ /\partial x$).

Commas between indices are usually superfluous and should be avoided if possible.

In a determinant use a notation which reduces it to the form $\det a_{ik}$.

Subscripts and superscripts cannot be printed in the same vertical column, hence the manuscript should be clear on whether a_i^k or $a^k{}_i$ is preferred. (Correspondingly, the limits of summation must not be typed *after* the Σ-sign, unless either Σ_1^m or $\Sigma^m{}_1$ is desired.) If a letter carrying a subscript has a prime, indicate whether x_i' or x'_i is desired.

Experience shows that a tilde or anything else *over* a letter is very unsatisfactory. Such symbols often drop out of the type after proof-reading and, when they do not, they usually appear uneven in print. For these reasons we advise against their use. This advice applies also to a bar over a Greek or German letter (for the symbol of complex conjugation an asterisk is often allowed by the context). Type carrying bars over ordinary size italic letters of the Latin alphabet is available.

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Repeated subscripts and superscripts should be used only when they cannot be avoided, since the index of the principal index usually appears about as large as the principal index. Bars and other devices *over* indices cannot be supplied. On the other hand, an asterisk or a prime (to be printed *after* the subscript) is possible on a subscript. The same holds true for superscripts.

Distinguish carefully between l. c. "oh," cap. "oh" and zero. One way of distinguishing them is by underlining one or two of them in different colors and explaining the meaning of the colors.

Distinguish between ϵ (epsilon) and $\in$ or ε (symbol), between x (eks) and $\times$ (multiplication sign), between l. c. and cap. phi, between l. c. and cap. psi, between l. c. k and kappa and between "ell" and "one" (for the latter, use 1 and l respectively).

Avoid unnecessary footnotes. For instance, references can be incorporated into the text (parenthetically, when necessary) by quoting the number in the bibliographic list, which appears at the end of the paper. Thus: "[3], pp. 261-266."

Except when informality in referring to papers or books is called for by the context, the following form is preferred:

[3] O. K. Blank, "Zur Theorie des Untermengenraumes der abstrakten Leermenge," *Bulletin de la Société Philharmonique de Zanzibar*, vol. 26 (1891), pp. 242-270.

In any case, the references should be precise, unambiguous and intelligible.

Usually sections numbers and section titles are printed in bold face, the titles "Theorem," "Lemma" and "Corollary" are in caps and small caps, "Proof," "Remark" and "Definition" are in italics. This (or a corresponding preference) should be marked in the manuscript. Use a period, and *not* a colon, after the titles Theorem, Lemma, etc.

German, script and bold face letters should be underlined in various colors and the meaning of the colors should be explained. The same device is needed for Greek letters if there is a chance of ambiguity. In general, mark all cap. Greek letters.

All instructions and explanations for the printer can conveniently be collected on a separate sheet, to be attached to the manuscript.

In case of doubt, recent issues of the Journal may be consulted.

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